EXAM C QUESTIONS OF THE WEEK

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A portfolio of insureds consists of two types of insureds. Losses from the three types are:
Type 1 insured loss: exponential with a mean of 1 ,
Type 2 insured loss: exponential with a mean of 2 ,
In the portfolio, $\frac{1}{2}$ of the insureds are of Type 1 and $\frac{1}{2}$ of the insureds are of Type 2.

Two insureds are chosen at random and one loss is observed from each insured.
The first insured is observed to have a loss of 1 and the second insured is observed to have a loss of 2. Find the probability that the two insured are of the same Type. Is it assumed that the losses of the two insureds are independent of one another.

Solution can be found below.

Week of September 18/06 - Solution

We will define $X_1$and X_2 to be the loss amounts from the first and second insured, respectively. We are given that $X_1 = 1$ and $X_2 = 2$.

We will define Λ to be the $\{1, 2\}$ random variable that identifies Type, and Λ_1 and Λ_2 are the types of insured 1 and insured 2.

We wish to find $P(\Lambda_1 = \Lambda_2 | X_1 = 1, X_2 = 2)$. Using the definition of conditional probability, this is $\frac{P[(\Lambda_1=\Lambda_2)\cap(X_1=1,X_2=2)]}{P(X_1=1,X_2=2)}$.

Since Λ must be 1 or 2, the numerator can be formulated as
$P[(\Lambda_1 = \Lambda_2) \cap (X_1 = 1, X_2 = 2)]$
$= P[(\Lambda_1 = \Lambda_2 = 1) \cap (X_1 = 1, X_2 = 2)] + P[(\Lambda_1 = \Lambda_2 = 2) \cap (X_1 = 1, X_2 = 2)]$
$= P(X_1 = 1, X_2 = 2 | \Lambda_1 = \Lambda_2 = 1) \cdot P(\Lambda_1 = \Lambda_2 = 1)$
$+ P(X_1 = 1, X_2 = 2 | \Lambda_1 = \Lambda_2 = 2) \cdot P(\Lambda_1 = \Lambda_2 = 2)$

Since both policies are chosen at random, $P(\Lambda_1 = \Lambda_2 = 1) = \frac{1}{4}$ and $P(\Lambda_1 = \Lambda_2 = 2) = \frac{1}{4}$.
Also, since the insureds are independent of one another,
$P(X_1 = 1, X_2 = 2 | \Lambda_1 = \Lambda_2 = 1)$
$= P(X_1 = 1 | \Lambda_1 = \Lambda_2 = 1) \cdot P(X_2 = 2 | \Lambda_1 = \Lambda_2 = 1) = e^{-1} \cdot e^{-2} = e^{-3}$
(the notation P really means density of X, not probability, in this situation).
Similarly, $P(X_1 = 1, X_2 = 2 | \Lambda_1 = \Lambda_2 = 2)$
$= P(X_1 = 1 | \Lambda_1 = \Lambda_2 = 2) \cdot P(X_2 = 2 | \Lambda_1 = \Lambda_2 = 1) = \frac{1}{2}e^{-1/2} \cdot \frac{1}{2}e^{-2/2} = \frac{1}{4}e^{-3/2}$.

Then, $P[(\Lambda_1 = \Lambda_2) \cap (X_1 = 1, X_2 = 2)] = e^{-3} \cdot \frac{1}{4} + \frac{1}{4}e^{-3/2} \cdot \frac{1}{4} = .026392$.

Also, $P(X_1 = 1, X_2 = 2) = P(X_1 = 1) \cdot P(X_2 = 2)$ because of independence of X_1 and X_2.
Each of the X_i's is a mixture of two exponenetials with pdf $\frac{1}{2}(e^{-x} + \frac{1}{2}e^{-x/2})$, so
$P(X_1 = 1) = \frac{1}{2}(e^{-1} + \frac{1}{2}e^{-1/2})$, and $P(X_1 = 2) = \frac{1}{2}(e^{-2} + \frac{1}{2}e^{-2/2})$, and
$P(X_1 = 1, X_2 = 2) = \frac{1}{2}(e^{-1} + \frac{1}{2}e^{-1/2}) \cdot \frac{1}{2}(e^{-2} + \frac{1}{2}e^{-2/2}) = .053570$.

Finally, $P(\Lambda_1 = \Lambda_2 | X_1 = 1, X_2 = 2) = \frac{.026392}{.053570} = .493$.