EXAM P QUESTIONS OF THE WEEK

S. Broverman, 2005

Question 8 - Week of September 12

According to the definition of the beta distribution X on the interval (0,1) with integer parameters $a \geq 1$ and $b \geq 1$, the pdf is $f(x) = \frac{(a+b-1)!}{(a-1)!(b-1)!} x^{a-1}(1-x)^{b-1}$.

Which of the following statements are true?

I. If $a = b$ then $E(X) = \frac{1}{2}$.

II. If $a = b$ then $Var(X) = \frac{1}{8a+1}$.

III. As k increases, $E(X^k)$ increases.

The solution can be found below.

Question 8 Solution

Since $f(x)$ is a pdf, it follows that $\int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{(a-1)!(b-1)!}{(a+b-1)!}$,

and this is valid for any integers a and b.

Then $E(X) = \int_0^1 x\,f(x)\,dx = \int_0^1 \frac{(a+b-1)!}{(a-1)!(b-1)!}x^a(1-x)^{b-1}dx$

$= \frac{(a+b-1)!}{(a-1)!(b-1)!}\int_0^1 x^a(1-x)^{b-1}dx = \frac{(a+b-1)!}{(a-1)!(b-1)!} \times \frac{a!(b-1)!}{(a+b)!} = \frac{a}{a+b}$

(this follows by using $a+1$ instead of a for the pdf).

Therefore, if $a = b$ then $E(X) = \frac{1}{2}$, so statement I is true.

$E(X^2) = \int_0^1 \frac{(a+b-1)!}{(a-1)!(b-1)!}x^{a+1}(1-x)^{b-1}dx = \frac{(a+b-1)!}{(a-1)!(b-1)!} \times \frac{(a+1)!(b-1)!}{(a+b+1)!}$

$= \frac{a(a+1)}{(a+b)(a+b+1)}$. If $a = b$ then $E(X^2) = \frac{a(a+1)}{(2a)(2a+1)} = \frac{a+1}{2(2a+1)}$.

Then with $a = b$ we have $Var(X) = \frac{a+1}{2(2a+1)} - (\frac{1}{2})^2 = \frac{a+1}{4a+2} - \frac{1}{4} = \frac{2}{4(4a+2)} = \frac{1}{8a+4}$.

Statement II is false.

$E(X^k) = \int_0^1 x^k\,f(x)\,dx = \int_0^1 \frac{(a+b-1)!}{(a-1)!(b-1)!}x^{a+k-1}(1-x)^{b-1}dx$.

Since $0 < x < 1$, as k increases x is raised to a higher power, so x^{a+k-1} becomes smaller numerically, and so does the integral. Statement III is false.