EXAM P QUESTIONS OF THE WEEK

S. Broverman, 2007

Week of September 17/07

An urn has 6 identically shaped balls. 4 of the balls are white and 2 of the balls are blue. A ball is chosen at random from the urn and replaced with a white ball. The procedure is done repeatedly. Find the probability that after the n-th application of this procedure there is one blue ball in the urn.

A) $(\frac{5}{6})^n + (\frac{2}{3})^n$

B) $(\frac{5}{6})^n - (\frac{2}{3})^n$

C) $2[(\frac{5}{6})^n + (\frac{2}{3})^n]$

D) $2[(\frac{5}{6})^n - (\frac{2}{3})^n]$

E) $2(\frac{5}{12})^n$

The solution can be found below.

Week of September 17/07 - Solution

In order for there to be one blue ball in the urn after the n application, it must be true that a blue ball was chosen exactly once in the n applications of the procedure. The blue ball could have been chosen on the 1st, or 2nd, . . . , or n-th application.

$P($blue ball chosen on 1st application and no blue ball chosen in next $n-1$ applications)
$= \frac{1}{3} \cdot (\frac{5}{6})^{n-1}$.

$P($1st blue ball chosen on 2nd application and no blue ball chosen in next $n-2$ applications)
$= \frac{2}{3} \cdot \frac{1}{3} \cdot (\frac{5}{6})^{n-2}$.

$\vdots$

$P($1st blue ball chosen on k-th application and no blue ball chosen in next $n-k$ applications)
$= (\frac{2}{3})^{k-1} \cdot \frac{1}{3} \cdot (\frac{5}{6})^{n-k}$.

$\vdots$

$P($1st blue ball chosen on n-th application) $= (\frac{2}{3})^{n-1} \cdot \frac{1}{3}$.

The probability in question is the sum of these:

$$\sum_{k=1}^{n} (\tfrac{2}{3})^{k-1} \cdot \tfrac{1}{3} \cdot (\tfrac{5}{6})^{n-k} = \tfrac{1}{3} \cdot (\tfrac{2}{3})^{-1} \cdot (\tfrac{5}{6})^{n} \cdot \sum_{k=1}^{n} (\tfrac{2}{3})^{k} (\tfrac{5}{6})^{-k}$$

$$= \tfrac{1}{2} \cdot (\tfrac{5}{6})^{n} \cdot \sum_{k=1}^{n} (\tfrac{2/3}{5/6})^{k} = \tfrac{1}{2} \cdot (\tfrac{5}{6})^{n} \cdot \sum_{k=1}^{n} (.8)^{k} = \tfrac{1}{2} \cdot (\tfrac{5}{6})^{n} \cdot (.8) \cdot \sum_{k=1}^{n} (.8)^{k-1}$$

$$= (.4) \cdot (\tfrac{5}{6})^{n} \cdot \tfrac{1-(.8)^{n}}{1-.8} = 2[(\tfrac{5}{6})^{n} - (\tfrac{2}{3})^{n}] \ .$$