EXAM P QUESTION OF THE WEEK

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The random variable X has pdf $f(x) = ae^{-x} + be^{-2x}$ for $x > 0$, where a and b are > 0.

You are also given that $E[X|X > 1] = 2$.

Find a and b.

The solution can be found below.

Week of March 31/08 - Solution

In order to be a pdf, we know that $\int_0^\infty f(x)\,dx = \int_0^\infty (ae^{-x} + be^{-2x})\,dx = a + \frac{b}{2} = 1$.

$P(X > 1) = \int_1^\infty (ae^{-x} + be^{-2x})\,dx = ae^{-1} + be^{-2}$.

The conditional pdf $f(x|X > 1)$ is $\frac{f(x)}{P(X>1)} = \frac{ae^{-x}+be^{-2x}}{ae^{-1}+be^{-2}}$ for $x > 1$.

Then, $E[X|X > 1] = \int_0^\infty x \cdot f(x|X > 1)\,dx = \frac{\int_1^\infty (axe^{-x}+bxe^{-2x})\,dx}{ae^{-1}+be^{-2}}$.

By integration by parts we get $\int_0^\infty xe^{-x}\,dx = -xe^{-x} - e^{-x}\Big|_{x=1}^{x=\infty} = 2e^{-1}$, and

$\int_0^\infty xe^{-2x}\,dx = -\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4}\Big|_{x=1}^{x=\infty} = \frac{3e^{-2}}{4}$.

Then, $E[X|X > 1] = \frac{2ae^{-1}+\frac{3be^{-2}}{4}}{ae^{-1}+be^{-2}} = 2$, so that $2ae^{-1} + \frac{3be^{-2}}{4} = 2ae^{-1} + 2be^{-2}$.

It follows from this equation that $b = 0$, and then from $a + \frac{b}{2} = 1$, we get $a = 1$.