EXAM P QUESTION OF THE WEEK

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Week of March 17/08

The random variable X has pdf $f(x) = e^{-x}$ for $x > 0$.

The random variable Y is defined as follows: $Y = \begin{cases} k & \text{if } k \text{ is even and } k < X \leq k+1 \\ -k & \text{if } k \text{ is odd and } k < X \leq k+1 \end{cases}$.

Find $E(Y)$.

The solution can be found below.

Week of March 17/08 - Solution

The probability function for Y is

$P(Y=k)=\int_k^{k+1} e^{-x}\,dx = e^{-k}-e^{-k-1}=e^{-k}(1-e^{-1})$ if k is even , and

$P(Y=-k)=\int_k^{k+1} e^{-x}\,dx = e^{-k}-e^{-k-1}=e^{-k}(1-e^{-1})$ if k is odd .

The expected value of Y is

$$\sum_{k=-\infty}^{\infty} k\cdot P(Y=k) = \sum_{k \text{ is even, } k>0}^{\infty} k\cdot P(Y=k) + \sum_{k \text{ is odd, } k>0}^{\infty} -k\cdot P(Y=k)$$

$$= \sum_{j=1}^{\infty} 2j\cdot P(Y=2j) - \sum_{j=1}^{\infty}(2j-1)\cdot P(Y=2j-1)$$
$$= \sum_{j=1}^{\infty} 2j\cdot e^{-2j}(1-e^{-1}) - \sum_{j=1}^{\infty}(2j-1)\cdot e^{-(2j-1)}(1-e^{-1})$$
$$= (1-e^{-1})\cdot [\ 2\cdot\sum_{j=1}^{\infty} j\cdot e^{-2j} - 2e\cdot\sum_{j=1}^{\infty} j\cdot e^{-2j} + e\cdot\sum_{j=1}^{\infty} e^{-2j}\,]\ .$$

Using the infinite geometric series $a+a^2+a^3+\cdots=\frac{a}{1-a}$
and $a+2a+3a^2+\cdots=\frac{a}{(1-a)^2}$, we get

$$E(Y)=(1-e^{-1})\cdot[2\cdot\tfrac{e^{-2}}{(1-e^{-2})^2} - 2e\cdot\tfrac{e^{-2}}{(1-e^{-2})^2} + e\cdot\tfrac{e^{-2}}{1-e^{-2}}]$$
$$=(1-e^{-1})\cdot\tfrac{2e^{-2}-e^{-1}-e^{-3}}{(1-e^{-2})^2} = (1-e^{-1})\cdot\tfrac{(-e^{-3})(e-1)^2}{[(1-e^{-1})(1+e^{-1})]^2} = \tfrac{1-e}{(e+1)^2}\ .$$