EXAM P QUESTIONS OF THE WEEK

S. Broverman, 2007

Week of August 13/07

X has a continuous uniform distribution on the interval $[0,1]$ and the conditional distribution of Y given $X = x$ is a continuous uniform distribution on the interval $[x, 2]$.

Find $E[Y]$.

A) $\frac{3}{4}$ B) 1 C) $\frac{5}{4}$ D) $\frac{3}{2}$ $\frac{7}{4}$

The solution can be found below.

Week of August 13/07 - Solution

The pdf of X is $f_X(x) = 1$ for $0 \le x \le 1$, and the conditional pdf of Y given $X = x$ is $f_{Y|X}(y|x) = \frac{1}{2-x}$ for $x \le Y \le 2$.

The joint density of X and Y is $f_{X,Y}(x,y) = f_{Y|X}(y|x) \cdot f_X(x) = \frac{1}{2-x}$ defined on the region $0 \le x \le 1$ and $x \le y \le 2$.

$E[Y] = \int_0^1 \int_x^2 y \cdot \frac{1}{2-x}\, dy\, dx = \int_0^1 \frac{4-x^2}{2(2-x)}\, dx = \int_0^1 \frac{2+x}{2}\, dx = \frac{5}{4}$.

An alternative, solution makes use of the rule $E[Y] = E[\,E[Y|X]\,]$.
Since the conditional distribution of $Y|X = x$ is uniform on the interval from x to 2, it follows that $E[Y|X] = \frac{X+2}{2}$. Then,
$E[Y] = E[\,E[Y|X]\,] = E[\frac{X+2}{2}] = \frac{1}{2}E[X] + 1$.
Since X is uniform on the interval from 0 to 1, $E[X] = \frac{1}{2}$.
Then, $E[Y] = (\frac{1}{2})(\frac{1}{2}) + 1 = \frac{5}{4}$.

Thanks to Gandalf for reminding me to include this approach to the solution.