EXAM C QUESTIONS OF THE WEEK

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Question 11 - Week of October 3

The distribution of the prior parameter λ is inverse gamma with parameters α and θ, where $\alpha > 2$. The distribution of the model random variable X is exponential with mean λ. For a particular (unknown) value of λ, n observed values of X are available, $x_1, ..., x_2$.

(a) Formulate the joint density of $x_1, ..., x_n$ and λ, and state the form of the posterior distribution of λ (indicate distribution type and parameter values). Show that the Bayesian premium (predictive expectation) can be written in the form $Z\overline{X} + (1-Z) \cdot \frac{\theta}{\alpha-1}$.

(b) Formulate the hypothetical mean, process variance, μ, a and v using the Buhlmann approach to credibility. Show that the Buhlmann credibility factor Z is the same as the factor Z in part (a) of this problem.

The solution can be found below.

Question 11 Solution

(a) $f(x_1,...,x_n,\lambda) = f(x_1|\lambda)\cdots f(x_n|\lambda)\cdot\pi(\lambda) = \frac{1}{\lambda^n}e^{-\Sigma x_i/\lambda}\cdot\frac{\theta^\alpha e^{-\theta/\lambda}}{\lambda^{\alpha+1}\Gamma(\alpha)}$

$= \frac{\theta^\alpha}{x^{\alpha+1}\Gamma(\alpha)}\cdot\frac{e^{-(\theta+\Sigma x_i)/\lambda}}{\lambda^{\alpha+n+1}}$.

From the form of the joint density, we see that the posterior distribution must be inverse gamma with parameters $\alpha' = \alpha + n$ and $\theta' = \theta + \Sigma x_i$.

The Bayesian premium is

$E[X_{n+1}|x_1,...,x_n] = \int_0^\infty E[X_{n+1}|\lambda]\cdot\pi(\lambda|x_1,...,x_n)\,d\lambda = \int_0^\infty \lambda\cdot\pi(\lambda|x_1,...,x_n)\,d\lambda$,

which is the mean of the posterior distribution, which is $\frac{\theta'}{\alpha'-1} = \frac{\theta+\Sigma x_i}{\alpha+n-1}$.

This can be written as $\frac{\theta+\Sigma x_i}{\alpha+n-1} = \frac{n\overline{x}}{\alpha+n-1} + \frac{\theta}{\alpha+n-1} = \frac{n}{\alpha+n-1}\cdot\overline{x} + (1-\frac{n}{\alpha+n-1})\cdot\frac{\theta}{\alpha-1}$,

so that $Z = \frac{n}{\alpha+n-1}$.

(b) Hypothetical mean is $\mu(\lambda) = E[X|\lambda] = \lambda$, process variance is $v(\lambda) = Var[X|\lambda] = \lambda^2$.

$\mu = E[\mu(\lambda)] = E[\lambda] = \frac{\theta}{\alpha-1}$,

$a = Var[\mu(\lambda)] = Var[\lambda] = E[\lambda^2] - (E[\lambda])^2 = \frac{\theta^2}{(\alpha-2)(\alpha-1)} - (\frac{\theta}{\alpha-1})^2 = \frac{\theta^2}{(\alpha-2)(\alpha-1)^2}$,

$v = E[v(\lambda)] = E[\lambda^2] = \frac{\theta^2}{(\alpha-2)(\alpha-1)}$. Then $\frac{v}{a} = \frac{\theta^2}{(\alpha-2)(\alpha-1)}\Big/\frac{\theta^2}{(\alpha-2)(\alpha-1)^2} = \alpha - 1$.

$Z = \frac{n}{n+\frac{v}{a}} = \frac{n}{n+\alpha-1}$. This is the same as Z in part (a)