

EXAM M QUESTIONS OF THE WEEK

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Week of May 8/06

A 3-year college names some students to the dean's honour list at the end of each year. The student might or might not be named to the list at the end of the 2nd and the end of the 3rd year. Some students also drop out each year. The college models student behaviour according to a non-homogeneous Markov Chain. At the start of each year, each student is classified as being in one of three states:

- 1 - not on the honour list at the end of the previous year,
- 2 - on the honour list at the end of the previous year, and
- 3 - dropped out before the end of the previous year.

Every new student starts out in state 1.

It is assumed that once a student drops out, the student does not return.

For the 3 years that the student is in college, we have the following one-step transition matrices:

$$\text{1st year } Q_0 = \begin{bmatrix} .5 & .25 & .25 \\ - & - & - \\ - & - & - \end{bmatrix}, \quad \text{2nd year } Q_1 = \begin{bmatrix} .6 & .2 & .2 \\ .4 & .5 & .1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\text{3rd year } Q_2 = \begin{bmatrix} .8 & .1 & .1 \\ .25 & .75 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The Dean's Honour List award pays \$1000 to each student that is named to the list at the end of each year. Find the total expected amount of scholarship that will be awarded to a new student over his 3 year college career from the Dean's Honour List Award.

The solution can be found below.

Week of May 8/06 - Solution

There is a payment of 1000 at time 1 with probability $Q_0^{(1,2)}$.

There is a payment of 1000 at time 2 with probability ${}_2Q_0^{(1,2)}$.

There is a payment of 1000 at time 3 with probability ${}_3Q_0^{(1,2)}$.

$$Q_0^{(1,2)} = .25.$$

$${}_2Q_0^{(1,2)} = Q_0^{(1,2)} \cdot Q_1^{(2,2)} + Q_0^{(1,1)} \cdot Q_1^{(1,2)} = (.25)(.5) + (.5)(.2) = .225.$$

Alternatively, ${}_2Q_0^{(1,2)}$ is the (1, 2)-entry in ${}_2Q = Q_0 \times Q_1$.

${}_3Q_0^{(1,2)}$ is the (1, 2)-entry in ${}_3Q = {}_2Q_0 \times Q_1$.

$${}_2Q_0 = \begin{bmatrix} .5 & .25 & .25 \\ - & - & - \\ - & - & - \end{bmatrix} \times \begin{bmatrix} .6 & .2 & .2 \\ .4 & .5 & .1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} .4 & .225 & .375 \\ - & - & - \\ - & - & - \end{bmatrix}$$

$${}_3Q_0 = \begin{bmatrix} .4 & .225 & .375 \\ - & - & - \\ - & - & - \end{bmatrix} \times \begin{bmatrix} .8 & .1 & .1 \\ .25 & .75 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} .37625 & .20875 & .415 \\ - & - & - \\ - & - & - \end{bmatrix}$$

$${}_3Q_0^{(1,2)} = .20875.$$

This can also be found by identifying paths from state 1 at time 0 to state 2 at time 3.

The expected amount of scholarship received by a student is

$$1000(.25 + .225 + .20875) = 683.75.$$