EXAM P QUESTIONS OF THE WEEK

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An insurer is considering insuring two independent risks. The loss for each risk has an exponential distribution with a mean of 1. The insurer is considering issuing two separate insurance policies, one for each risk, each of which has a policy limit (maximum payment) of 2. The insurer is also considering issuing a single policy covering the combined loss on both risks, with a policy limit of 4. We denote by A the expected insurance payment for each of the two separate policies, and we denote by B the expected insurance payment for the single policy covering the combined loss. Find B/A.

The solution can be found below.

Week of July 31/06 - Solution

Suppose that X is an exponential random variable with mean 1.
The pdf of X is $f_X(x) = e^{-x}$, $x > 0$. The insurance policy for a single risk with policy limit 2 will pay $\begin{cases} x & \text{if } 0 < x \leq 2 \\ 2 & \text{if } x > 2 \end{cases}$.
The expected amount paid for one policy is

$$A = \int_0^2 x \cdot f_X(x)\,dx + 2 \cdot P(X > 2) = \int_0^2 x \cdot e^{-x}\,dx + 2 \cdot e^{-2}$$

$$= (-xe^{-x} - e^{-x})\Big|_{x=0}^{x=2} + 2e^{-2} = (-2e^{-2} - e^{-2}) - (0-1) + 2e^{-2} = 1 - e^{-2} \text{ .}$$

Note that, for a non-negative random variable $X \geq 0$, with a policy limit u, the expected insurance payment can also be formulated as $\int_0^u [1 - F_X(x)]\,dx$. In the case of the exponential distribution with mean 1, $F_X(x) = 1 - e^{-x}$, so the expected insurance payment with a policy limit of 2 is $\int_0^2 [1 - (1 - e^{-x})]\,dx = \int_0^2 e^{-x}\,dx = 1 - e^{-2}$. $A = 2(1 - e^{-2})$.

Suppose that X_1 and X_2 are the independent exponential losses on the two risks. The combined loss is $Y = X_1 + X_2$, and the insurance on the combined losses will apply a limit of 4 to Y.
The sum of two independent exponential random variables, each with a mean of 1, is a gamma random variable with pdf $f_Y(y) = ye^{-y}$, $y > 0$. This can be verified a couple of ways.
(i) Convolution:

$$f_Y(y) = \int_0^y f_{X_1}(x) \cdot f_{X_2}(y - x)\,dx = \int_0^y e^{-x} \cdot e^{-(y-x)}\,dx = \int_0^y e^{-y}\,dx = ye^{-y}$$

(ii) Transformation of random variables:
Since X_1 and X_2 are independent, the joint distribution of X_1 and X_2 has pdf

$$f(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) = e^{-x_1} \cdot e^{-x_2}$$

$U = X_1$, $Y = X_1 + X_2 \rightarrow X_1 = U$, $X_2 = Y - U$

$\rightarrow$ pdf of U, Y is $g(u, y) = f(u, y - u) \cdot \begin{vmatrix} \frac{\partial}{\partial u} u & \frac{\partial}{\partial u} y - u \\ \frac{\partial}{\partial y} u & \frac{\partial}{\partial y} y - u \end{vmatrix} = e^{-u} \cdot e^{-(y-u)} \cdot 1 = e^{-y}$,

and the joint distribution of U and Y is defined on the region $0 < u < y$
(this is true because $u = x_1 < x_1 + x_2 = y$).
The marginal density of Y is $f_Y(y) = \int_0^y g(u, y)\,du = \int_0^y e^{-y}\,dy = ye^{-y}$.

We impose a limit of 4 for the insurance policy on Y, the combination of the two exponential losses. The amount paid by the insurance is $\begin{cases} y & \text{if } 0 < y \le 4 \\ 4 & \text{if } y > 4 \end{cases}$

The expected insurance payment is $\int_0^4 y \cdot f_Y(y)\, dy + 4 \cdot P(Y > 4)$.

$\int_0^4 y \cdot f_Y(y)\, dy = \int_0^4 y \cdot ye^{-y}\, dy = \int_0^4 y^2 \cdot e^{-y}\, dy$.
Applying integration by parts, this becomes
$-y^2 e^{-y}\Big|_{y=0}^{y=4} - \int_0^4 (-e^{-y})(2y)\, dy = -16e^{-4} + 2\int_0^4 ye^{-y}\, dy$
$= -16e^{-4} + 2 \cdot [-ye^{-y} - e^{-y}\Big|_{y=0}^{y=4}] = -16e^{-4} + 2[-4e^{-4} - e^{-4} - (0-1)]$
$= 2 - 26e^{-4}$.

$P(Y > 4) = \int_4^\infty f_Y(y)\, dy = \int_4^\infty ye^{-y}\, dy = (-ye^{-y} - e^{-y})\Big|_{y=4}^{y=\infty}$
$= (-0-0) - (-4e^{-4} - e^{-4}) = 5e^{-4}$.

Expected insurance payment of the combined policy is $2 - 26e^{-4} + 4(5e^{-4}) = 2 - 6e^{-4} = B$.

The ratio B/A is $\frac{2-6e^{-4}}{2(1-e^{-2})} = 1.093$.