EXAM P QUESTIONS OF THE WEEK

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Week of January 9/06

The number of streetcars passing Grant Park station during a 10-minute period has a Poisson distribution with a mean of λ. A streetcar stops at Grant Park station only if someone is either leaving the streetcar or boarding the streetcar at Grant Park station. According to the model used by the transit system, on any given streetcar, the number of people who will leave the streetcar at Grant Park station has a Poisson distribution with a mean of 2 and the number of people who will board the streetcar at Grant Park station has a Poisson distribution with a mean of 1. The transit model assumes that the number of people boarding at Grant Park station and the number of people leaving at Grant Park station are independent of one another, and are independent of the numbers boarding and leaving any previous streetcars. The transit system would like to determine the value of λ so that the probability of no passengers leaving or boarding a streetcar at Grant park station during a 10 minute period is .02 . Find λ.

The solution can be found below.

Week of January 9/06 - Solution

If order for there to be no passengers leaving or boarding a streetcar in a 10-minute period, one of the following must happen:

A_0 : no streetcars arrive during the 10-minute period,

A_1 : 1 street car arrives during the 10-minute period and no one boards or leaves,

A_2 : 2 street cars arrives during the 10-minute period and no one boards or leaves either,

A_3 : 3 street cars arrives during the 10-minute period and no one boards or leaves any of them,

. . . ,

A_n : n street cars arrives during the 10-minute period and no one boards or leaves any of them,

. . .

The overall probability is $P(A_0) + P(A_1) + P(A_2) + \cdots$.

We will denote by N the number of streetcars reaching Grant Park station in a 10-minute period. We will denote by B the number of people boarding a particular streetcar at Grant Par station, and we will denote by L the number of people leaving a particular streetcar at Grant park station.

$P(A_0) = P(N = 0) = e^{-\lambda}$ (the probability of 0 streetcars reaching Grant Park station in a 10-minute period).

$P(A_1) = P[(N = 1) \cap (B = 0) \cap (L = 0)] = \lambda e^{-\lambda} \cdot e^{-1} \cdot e^{-2} = \lambda e^{-\lambda} \cdot e^{-3}$.

$P(A_2) = P[(N = 2) \cap (B_1 = 0) \cap (B_2 = 0) \cap (L_1 = 0) \cap (L_2 = 0)]$
$= \frac{\lambda^2 e^{-\lambda}}{2!} \cdot e^{-1} \cdot e^{-1} \cdot e^{-2} \cdot e^{-2} = \frac{\lambda^2 e^{-\lambda}}{2!} \cdot (e^{-3})^2$.

B_1 is the number boarding the first streetcar and B_2 is the number boarding the second streetcar, and similarly L_1 and L_2 are the numbers leaving the first and second streetcars. We then use independence of those events to get the probability.

$P(A_3) = P[(N = 3) \cap (B_1 = 0) \cap (B_2 = 0) \cap (B_3 = 0) \cap (L_1 = 0) \cap (L_2 = 0) \cap (L_3 = 0)]$
$= \frac{\lambda^3 e^{-\lambda}}{3!} \cdot (e^{-1})^3 \cdot (e^{-2})^3 = \frac{\lambda^3 e^{-\lambda}}{3!} \cdot (e^{-3})^3$.

In general, we can see that

$P(A_n) = P[(N = n) \cap (B_1 = 0) \cap \cdots \cap (B_n = 0) \cap (L_1 = 0) \cap \cdots \cap (L_n = 0)]$
$= \frac{\lambda^n e^{-\lambda}}{n!} \cdot (e^{-1})^n \cdot (e^{-2})^n = \frac{\lambda^n e^{-\lambda}}{n!} \cdot (e^{-3})^n = e^{-\lambda} \cdot \frac{(\lambda e^{-3})^n}{n!}$ for $n = 0, 1, 2, \ldots$

The overall probability is

$$P(A_0) + P(A_1) + P(A_2) + \cdots = \sum_{n=0}^{\infty} P(A_n) = \sum_{n=0}^{\infty} e^{-\lambda} \cdot \frac{(\lambda e^{-3})^n}{n!} = e^{-\lambda} \cdot \sum_{n=0}^{\infty} \frac{(\lambda e^{-3})^n}{n!} .$$

Recalling the exponential Taylor series $1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$, the overall probability becomes $e^{-\lambda} \cdot \sum_{n=0}^{\infty} \frac{(\lambda e^{-3})^n}{n!} = e^{-\lambda} \cdot e^{-\lambda e^{-3}} = e^{-\lambda(1+e^{-3})}$.

In order for this to be .1, we must have $ln(.02) = -\lambda(1 + e^{-3})$, so that $\lambda = \frac{-ln(.02)}{1+e^{-3}} = 3.726$.