EXAM C QUESTIONS OF THE WEEK

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Week of January 30/06

The random variable X is uniformly distributed on the interval $(2\theta\,,\,3\theta)$ where $\theta > 0$.
$X_1, ..., X_n$ is a random sample from the distribution of X.

(a) Show that the mle of θ is $\widehat{\theta} = \frac{1}{3} \cdot Max(X_1\,,\,\ldots\,,\,X_n)$.

(b) Show that $\widehat{\theta}$ is asymptotically unbiased.

(c) Find the variance of $\widehat{\theta}$ if $n = 2$.

Solution can be found below.

<u>Week of January 30/06 - Solution</u>

(a) The pdf of X is $f(x) = \frac{1}{\theta}$ for $2\theta < x < 3\theta$, so the likelihood function is
$L(\theta) = \prod_{i=1}^{n} f(x_i) = \frac{1}{\theta^n}$. $L(\theta)$ will be maximized when θ is minimized.
For each x_i it must be true that $\frac{X_i}{3} < \theta < \frac{X_i}{2}$. The smallest possible value of θ that is consistent with these inequalities <u>for all x_i sample values</u> is
$\widehat{\theta} = Max(\frac{X_1}{3}, \frac{X_2}{3}, \ldots, \frac{X_n}{3}) = \frac{1}{3} \cdot Max(X_1, \ldots, X_n)$. This is the mle.

(b) The bias in the estimator $\widehat{\theta}$ is $E[\widehat{\theta}] - \theta$.
To find $E[\widehat{\theta}]$ we first find the cdf of $\widehat{\theta}$, $F_{\widehat{\theta}}(t) = P[\widehat{\theta} \le t]$.
This is $P[\widehat{\theta} \le t] = P[\frac{1}{3} \cdot Max(X_1, \ldots, X_n) \le t] = P[Max(X_1, \ldots, X_n) \le 3t]$
$= P[(X_1 \le 3t) \cap (X_2 \le 3t) \cap \cdots \cap (X_n \le 3t)] = [P(X \le 3t)]^n$
(since the X_i's are mutually independent).
For $t < \frac{2\theta}{3}$, we have $3t < 2\theta$, so that $P(X \le 3t) = 0$,
and for $t > \theta$, we have $3t > 3\theta$, so that $P(X \le 3t) = 1$.
For $\frac{2\theta}{3} \le t \le \theta$ we have $2\theta \le 3t \le 3\theta$, so that $P(X \le 3t) = \frac{3t - 2\theta}{\theta}$
(X has a uniform distribution from 2θ to 3θ).
Therefore, for $\frac{2\theta}{3} \le t \le \theta$ we have $F_{\widehat{\theta}}(t) = P[\widehat{\theta} \le t] = [P(X \le 3t)]^n = [\frac{3t-2\theta}{\theta}]^n$.
Since $\theta > 0$, the X_i's are all > 0 , and $\widehat{\theta} > 0$. Therefore,
$E[\widehat{\theta}] = \int_0^{\infty} [1 - F_{\widehat{\theta}}(t)]\, dt = \int_0^{2\theta/3} [1 - 0]\, dt + \int_{2\theta/3}^{\theta} (1 - [\frac{3t-2\theta}{\theta}]^n)\, dt + \int_{\theta}^{\infty} (1 - 1)\, dt$
$= \int_0^{\theta} 1\, dt - \int_{2\theta/3}^{\theta} [\frac{3t-2\theta}{\theta}]^n\, dt = \theta - \left[\frac{(3t-2\theta)}{3(n+1)\theta^n}^{n+1} \Big|_{t=2\theta/3}^{t=\theta} \right] = \theta - \frac{\theta}{3(n+1)} = (\frac{3n+2}{3n+3})\theta$.
Then the bias in the estimator $\widehat{\theta}$ is $B(\widehat{\theta}) = E[\widehat{\theta}] - \theta = (\frac{3n+2}{3n+3})\theta - \theta = -\frac{\theta}{3n+3}$.
We see that $\lim_{n \to \infty} B(\widehat{\theta}) = 0$, so $\widehat{\theta}$ is asymptotically unbiased.

(c) The variance of $\widehat{\theta}$ is $Var(\widehat{\theta}) = E[\widehat{\theta}^2] - (E[\widehat{\theta}])^2$.
We have just seen that $E[\widehat{\theta}] = (\frac{3n+2}{3n+3})\theta$.
If $n = 2$, we have $E[\widehat{\theta}] = \frac{8\theta}{9}$.
From the cdf of $\widehat{\theta}$, $F_{\widehat{\theta}}(t) = [\frac{3t-2\theta}{\theta}]^2$, we can get the pdf of $\widehat{\theta}$,
$f_{\widehat{\theta}}(t) = \frac{d}{dt} F_{\widehat{\theta}}(t) = 2 \cdot [\frac{3t-2\theta}{\theta}] \cdot \frac{3}{\theta}$.
Then,
$E[\widehat{\theta}^2] = \int_{2\theta/3}^{\theta} t^2 \cdot f_{\widehat{\theta}}(t)\, dt = \int_{2\theta/3}^{\theta} t^2 \cdot 2 \cdot [\frac{3t-2\theta}{\theta}] \cdot \frac{3}{\theta}\, dt$
$= \frac{6}{\theta^2} \cdot \int_{2\theta/3}^{\theta} (3t^3 - 2\theta t^2)\, dt = \frac{6}{\theta^2} \cdot \frac{43\theta^4}{324} = \frac{43\theta^2}{54}$.
$Var[\widehat{\theta}] = E[\widehat{\theta}^2] - (E[\widehat{\theta}])^2 = \frac{43\theta^2}{54} - (\frac{8\theta}{9})^2 = \frac{\theta^2}{162}$.