EXAM P QUESTIONS OF THE WEEK

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Week of January 2/06

The continuous random variable X is defined on the interval $(0,1)$.
For each number a in $(0,1)$, you are given the conditional expectation
$E[\,X\,|\,X \leq a\,] \,=\, \frac{3a}{4}$.
If $0 < a < 1$, find $E[\,X\,|\,X > a\,]$.

Hint: You may need to use the following version of Leibniz's rule:
$\frac{d}{dz}\int_0^z h(x,z)\,dx \,=\, h(z,z) + \int_0^z \left[\frac{d}{dz}\,h(x,z)\right] dx$.

The solution can be found below.

Week of January 2/06 - Solution

The conditional pdf of X given $X \le a$ is $\frac{f(x)}{F(a)}$, and the conditional expectation is
$E[\,X\,|\,X \le a\,] \;=\; \int_0^a x \cdot \frac{f(x)}{F(a)}\,dx \;=\; \frac{3a}{4}$.
$\frac{d}{da}\,E[\,X\,|\,X \le a\,] \;=\frac{3}{4} =\; \frac{d}{da}\int_0^a x \cdot \frac{f(x)}{F(a)}\,dx = \frac{af(a)}{F(a)} + \int_0^a \frac{d}{da}[x \cdot \frac{f(x)}{F(a)}]\,dx$.

Since $\frac{d}{da}[x \cdot \frac{f(x)}{F(a)}] \;=\; -\frac{xf(x)f(a)}{(F(a))^2}$, we get
$\int_0^a \frac{d}{da}[x \cdot \frac{f(x)}{F(a)}]\,dx = \; -\int_0^a \frac{xf(x)f(a)}{(F(a))^2}\,dx = \; -\frac{f(a)}{F(a)} \cdot \int_0^a x \cdot \frac{f(x)}{F(a)}\,dx$
$\qquad = \; -\frac{f(a)}{F(a)} \cdot E[\,X\,|\,X \le a\,] \;=\; -\frac{f(a)}{F(a)} \cdot \frac{3a}{4}$.

Therefore, $\frac{3}{4} = \frac{af(a)}{F(a)} - \frac{f(a)}{F(a)} \cdot \frac{3a}{4} = \frac{af(a)}{4F(a)}$, so that $\frac{f(a)}{F(a)} = \frac{3}{a}$.
Noting that $\frac{f(a)}{F(a)} = \frac{d}{da}\,ln\,F(a)$, we have $\frac{d}{da}\,ln\,F(a) = \frac{3}{a}$, and
$ln\,F(a) = 3\,ln\,a \,+\, c$, and $F(a) = a^3 \cdot e^c$ (where c is a constant).
Since X is defined on the interval $(0,1)$, we have $F(1) = 1 = e^c$, so that $c = 0$.
Therefore, $F(a) = a^3$ and $f(x) = F'(x) = 3x^2$.

Then $E[X|X > a] = \int_a^1 x \cdot \frac{f(x)}{1-F(a)}\,dx = \int_a^1 x \cdot \frac{3x^2}{1-a^3}\,dx = \frac{3}{4} \cdot \frac{1-a^4}{1-a^3}$.

Alternatively, we can use the rule
$E[X] = E[X|X \le a] \cdot P(X \le a) + E[X|X > a] \cdot P(X > a)$
$\qquad = E[X|X \le a] \cdot F(a) + E[X|X > a] \cdot [1 - F(a)]$.
From the given conditional expectation $E[\,X\,|\,X \le a\,] \;=\; \frac{3a}{4}$, we get
$E[X] = E[\,X\,|\,X \le 1\,] \;=\; \frac{3}{4}$.
Once we have found $F(a) = a^3$, we get
$\frac{3}{4} = E[X] = E[X|X \le a] \cdot F(a) + E[X|X > a] \cdot [1 - F(a)]$
$\qquad = \frac{3a}{4} \cdot a^3 + E[X|X > a] \cdot (1 - a^3)$,
and solving for $E[X|X > a]$ results in $E[X|X > a] = \frac{3}{4} \cdot \frac{1-a^4}{1-a^3}$.