EXAM FM QUESTIONS OF THE WEEK

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Week of January 2/06

Today is the first day of the month, and it is Smith's 40th birthday, and he has just started a new job today. He will receive a paycheck at the end of each month (starting with this month). His salary will increase by 3% every year (his monthly paychecks during a year are level), with the first increase occurring just after his 41st birthday. He wishes to take c% of each paycheck and deposit that amount into an account earning interest at an annual effective rate of 5%. Just after the deposit on the day before his 65th birthday, Smith uses the full balance in the account to purchase a 15-year annuity. The annuity will make monthly payments starting at the end of the month of Smith's 65th birthday. The monthly payments will be level during each year, and will increase by 5% every year (with the first increase occurring in the year Smith turns 66). The starting monthly payment when Smith is 65 will be 50% of Smith's final monthly salary payment. Find c.

The solution can be found below.

Week of January 2/06 - Solution

Suppose that Smith's starting monthly salary when he is 40 is K. The monthly deposit in the first year is $.01cK$, and the accumulated value of the monthly deposits at the end of the first year is $X = .01cKs_{\overline{12}|j}$, where $(1+j)^{12} = 1.05$ (j is the equivalent monthly interest rate). This is the equivalent single deposit that Smith would have to make just before his 41st birthday in order to have the same amount in his account as he would have from the year's monthly deposits.
Smith's salary increases by 3% when he is 41, so in the year that Smith is 41, the accumulated value at the end of the year of that year's monthly deposits is $(1.03)(.01cKs_{\overline{12}|j}) = 1.03X$.
In the year that Smith is 42, the accumulated value at the end of the year of that year's monthly deposits is $(1.03)^2(.01cKs_{\overline{12}|j}) = (1.03)^2 \cdot X$.
This pattern continues during the period Smith makes the deposits. The following time line illustrates the series of equivalent annual deposits. The reason that we consider equivalent annual deposits is that the geometric frequency for salary increases is once per year, so in order to use the standard formula for an annuity with geometric payments, we must have the payment period coincide with the geometric growth period.

Time	0	1	2	3	4	...	24	25
Age (just after deposit)	40	41	42	43	44	...	64	65
Equiv. Deposit		X	$1.03X$	1.03^2X	1.03^3X	...	$1.03^{23}X$	$1.03^{24}X$

The accumulated value of the deposits at the time of the final deposit is
$X \cdot \frac{(1.05)^{25}-(1.03)^{25}}{.05-.03} = .01cKs_{\overline{12}|j} \cdot \frac{(1.05)^{25}-(1.03)^{25}}{.05-.03} = 7.931625cK$.

We want this amount to be equal to the present value of the 15-year annuity whose first payment is one month after the last deposit. The monthly payment in the first year will be 50% of Smith's monthly salary during the year he was 64. That monthly salary was $1.03^{24} \cdot K$, so the first year's monthly annuity payment is $\frac{1}{2} \cdot (1.03)^{24} \cdot K = 1.016397K$. The annuity payment is level for the year, but grows by 5% every year. Again, we look at an equivalent annual payment at the end of each year. Since the interest rate is still annual effective 5%, the equivalent annual payment at the end of the first year of the annuity is $1.016397Ks_{\overline{12}|j} = 12.473810K = Y$. The subsequent 14 equivalent annual annuity payments are $1.05Y$, 1.05^2Y , $\ldots$, $1.05^{14}Y$.
Since the annual effective interest rate is 5%, which is equal to the annual geometric growth rate of the annuity, the present value of Smith's annuity, one month before the first monthly payment occurs (whish is the same as one year before the first equivalent annual payment occurs) is $15vY = 15 \cdot \frac{Y}{1.05} = 178.197286K$.

Setting this equal to the accumulated value of Smith's deposits, we get
$7.931625cK = 178.197286K$, so that $c = 22.5$.