EXAM M QUESTIONS OF THE WEEK

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Week of February 27/06

X is a continuous non-negative random variable with a finite mean and infinite support (defined on $(0,\infty)$) and with pdf $f(x)$ and cdf $F(x)$. Two new random variables, Y and Z are defined related to X.

Y has cdf $F_Y(y) = \frac{\int_0^y [1-F(x)]\,dx}{E[X]}$ and

Z has cdf $F_Z(z) = \frac{\int_0^z x\cdot f(x)\,dx}{E[X]}$.

You are given that Y and Z have similar (proportional) right tails.
Which of the following could be the distribution of X?
I. Exponential distribution II. Pareto distribution

The solution can be found below.

Week of February 27/06 - Solution

We are given that $\lim_{t\to\infty} \frac{S_Y(t)}{S_Z(t)} = c$, where $0 < c < \infty$
(this is the definition Y and Z having similar right tails).

$$\frac{S_Y(t)}{S_Z(t)} = \frac{1-F_Y(t)}{1-F_Z(t)} = \frac{1-\frac{\int_0^t [1-F(x)]\,dx}{E[X]}}{1-\frac{\int_0^t x\cdot f(x)\,dx}{E[X]}} = \frac{E[X]-\int_0^t [1-F(x)]\,dx}{E[X]-\int_0^t x\cdot f(x)\,dx}.$$

When limit is taken as $t\to\infty$, the numerator and denominator both approach 0, so we apply l'Hôspital's rule to take the limit. According to l'Hôspital's rule, we differentiate with respect to t both the numerator and denominator, and then take the limit of the ratio. The ratio is
$\frac{1-F(t)}{t\cdot f(t)} = \frac{S(t)}{t\cdot f(t)} = \frac{1}{t\cdot h(t)}$, where $h(t)$ is the hazard function of X.
Therefore $\lim_{t\to\infty} \frac{1}{t\cdot h(t)} = c$.

Suppose that X has an exponential distribution. Then $f(x) = \frac{e^{-x/\theta}}{\theta}$, and $S(x) = e^{-x/\theta}$,
so that $h(t) = \frac{f(t)}{S(t)} = \frac{1}{\theta}$. Then, $\lim_{t\to\infty} \frac{1}{t\cdot h(t)} = \frac{\theta}{t} = 0$
Therefore, X cannot have an exponential distribution.

Suppose that X has a Pareto distribution with parameters α and θ. Then $f(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}$
and $S(x) = (\frac{\theta}{x+\theta})^\alpha$, so that $h(t) = \frac{f(t)}{S(t)} = \frac{\alpha}{t+\theta}$.
Then, $\lim_{t\to\infty} \frac{1}{t\cdot h(t)} = \lim_{t\to\infty} \frac{t+\theta}{\alpha t} = \frac{1}{\alpha}$.
For the Pareto distribution, $0 < \alpha < \infty$, and it follows that Y and Z have similar right tails.
X can have a Pareto distribution.