EXAM M QUESTIONS OF THE WEEK

S. Broverman, 2006

Week of February 20/06

A portfolio of risks models the annual loss of an individual risk as having an exponential distribution with a mean of Λ. For a randomly selected risk from the portfolio, the value of Λ has an inverse gamma distribution with a mean of 20 and a standard deviation of 10. For a randomly chosen risk, find the probability that the annual loss for that risk is greater than 20.

The solution can be found below.

Week of February 20/06 - Solution

Given $\Lambda = \lambda$, the annual loss X has an exponential distribution with mean λ and Λ has an inverse gamma distribution. X is a continuous mixture distribution of an "exponential over an inverse gamma".

Suppose that the inverse gamma distribution of Λ has parameters α and θ. We are given that the mean and standard deviation of Λ are 20 and 10. Therefore, the variance of Λ is 100 and the 2nd moment of Λ is $E[\Lambda^2] = Var[\Lambda] + (E[\Lambda])^2 = 10^2 + 20^2 = 500$.
The mean of an inverse gamma is $\frac{\theta}{\alpha-1}$. and the 2nd moment is $\frac{\theta^2}{(\alpha-2)(\alpha-1)}$.
From the two equations $\frac{\theta}{\alpha-1} = 20$ and $\frac{\theta^2}{(\alpha-2)(\alpha-1)} = 500$,
we get $\frac{500}{20^2} = \frac{\theta^2}{(\alpha-2)(\alpha-1)} \Big/ (\frac{\theta}{\alpha-1})^2 = \frac{\alpha-1}{\alpha-2}$. Then solving for α results in $\alpha = 6$.
Substituting back into $\frac{\theta}{\alpha-1} = 20$, we get that $\theta = 100$.

When we have a continuous mixture distribution for X over Λ, the pdf, expected values and probabilities for the marginal distribution of X can be found by conditioning over Λ.
The conditional pdf of X given $\Lambda = \lambda$ is $f(x|\Lambda = \lambda) = \frac{1}{\lambda} e^{-x/\lambda}$ and the pdf of the inverse gamma distribution of Λ is $f_\Lambda(\lambda) = \frac{\theta^\alpha e^{-\theta/\lambda}}{\lambda^{\alpha+1} \cdot \Gamma(\alpha)}$. From the calculated parameter values, we have $f_\Lambda(\lambda) = \frac{100^6 e^{-100/\lambda}}{\lambda^7 \cdot \Gamma(6)}$.

We can find the probability $P(X > 20)$ by conditioning over λ:
$P(X > 20) = \int_0^\infty P(X > 20|\lambda) \cdot f_\Lambda(\lambda)\, d\lambda = \int_0^\infty e^{-20/\lambda} \cdot \frac{100^6 e^{-100/\lambda}}{\lambda^7 \cdot \Gamma(6)}\, d\lambda = \frac{100^6}{\Gamma(6)} \cdot \int_0^\infty \frac{e^{-120/\lambda}}{\lambda^7}\, d\lambda$

We know that the inverse gamma pdf must integrate to 1 (as any pdf must), so that $\int_0^\infty \frac{\theta^\alpha e^{-\theta/\lambda}}{\lambda^{\alpha+1} \cdot \Gamma(\alpha)}\, d\lambda = 1$. It follows that $\int_0^\infty \frac{e^{-\theta/\lambda}}{\lambda^{\alpha+1}}\, d\lambda = \frac{\Gamma(\alpha)}{\theta^\alpha}$.
Therefore, $\int_0^\infty \frac{e^{-120/\lambda}}{\lambda^7}\, d\lambda = \int_0^\infty \frac{e^{-120/\lambda}}{\lambda^{6+1}}\, d\lambda = \frac{\Gamma(6)}{120^6}$, and
$P(X > 20) = \frac{100^6}{\Gamma(6)} \cdot \frac{\Gamma(6)}{120^6} = .335$.

It is possible to show in general that if the conditional distribution of X given Λ is exponential with mean Λ, and if Λ has an inverse gamma distribution with parameters α and θ, then the marginal (unconditional) distribution of X is (two-parameter) Pareto with parameters α and θ.
In this example, the marginal distribution of X would be Pareto with parameters $\alpha = 6$ and $\theta = 100$, and $P(X > x) = (\frac{\theta}{\theta+x})^\alpha$, so that $P(X > 20) = (\frac{100}{120})^6$.