EXAM C QUESTION OF THE WEEK

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The method of moments is applied to estimate the mixing weights and Poisson parameters for a mixture of two Poisson distributions. The estimates of the first three moments of the mixed distribution are:

- estimated first moment is 3
- estimated second moment is 14
- estimated third moment is 83

Determine the estimated mixing weights and Poisson means.

The solution can be found below.

Week of March 24/08 - Solution

The second moment of a Poisson distribution with mean λ is $\lambda^2 + \lambda$.

The third moment can be found from the moment generating function of the Poisson distribution, $M(t) = e^{\lambda(e^t - 1)}$, so that

$\frac{d}{dt^3} M(t)\Big|_{t=0} = e^{2(\lambda+1)(e^t-1)} \cdot [\lambda^3 e^{3t} + 3\lambda^2 e^{2t} + \lambda e^t] = \lambda^3 + 3\lambda^2 + \lambda.$

The mixing weights will be denoted a and $1 - a$ for Poisson with means λ_1 and λ_2 respectively.

We get the following three equations from the given information:

$a\lambda_1 + (1-a)\lambda_2 = 3$

$a(\lambda_1^2 + \lambda_1) + (1-a)(\lambda_2^2 + \lambda_2) = 14$

$a[\lambda_1^3 + 3\lambda_1^2 + \lambda_1] + (1-a)[\lambda_2^3 + 3\lambda_2^2 + \lambda_2] = 83$

From the first equation we get $a = \frac{\lambda_2 - 3}{\lambda_2 - \lambda_1}$. Applying this to equation 2 results in the equation

$\frac{\lambda_2-3}{\lambda_2-\lambda_1} \cdot (\lambda_1^2 + \lambda_1) + \frac{3-\lambda_1}{\lambda_2-\lambda_1} \cdot (\lambda_2^2 + \lambda_2) = 14$.

This equation can be written in the form $\frac{3[\lambda_2^2+\lambda_2-(\lambda_1^2+\lambda_1)]-\lambda_1(\lambda_2^2+\lambda_2)+\lambda_2(\lambda_1^2+\lambda_1)}{\lambda_2-\lambda_1} = 14$.

This reduces to $\frac{3[(\lambda_2-\lambda_1)^2+\lambda_2-\lambda_1]-\lambda_1\lambda_2[\lambda_2-\lambda_1]}{\lambda_2-\lambda_1} = 14$,

and after cancellation, the equation becomes $3[(\lambda_2 + \lambda_1) + 1] - \lambda_1\lambda_2 = 14$.

Applying $a = \frac{\lambda_2-3}{\lambda_2-\lambda_1}$ to equation 3 results in the equation

$\frac{\lambda_2-3}{\lambda_2-\lambda_1} \cdot [\lambda_1^3 + 3\lambda_1^2 + \lambda_1] + \frac{3-\lambda_1}{\lambda_2-\lambda_1} \cdot [\lambda_2^3 + 3\lambda_2^2 + \lambda_2] = 83$.

Regrouping terms in this equation, we get

$\frac{3[\lambda_2^3-\lambda_1^3+3\lambda_2^2-3\lambda_1^2+\lambda_2-\lambda_1]}{\lambda_2-\lambda_1} - \frac{\lambda_1\lambda_2[\lambda_2^2+3\lambda_2-\lambda_1^2-3\lambda_1]}{\lambda_2-\lambda_1} = 83$.

This reduces to

$\frac{3[(\lambda_2-\lambda_1)(\lambda_2^2+\lambda_1\lambda_2+\lambda_1^2)+3(\lambda_2-\lambda_1)(\lambda_2+\lambda_1)+\lambda_2-\lambda_1]}{\lambda_2-\lambda_1} - \frac{\lambda_1\lambda_2[(\lambda_2-\lambda_1)(\lambda_2+\lambda_1)+3(\lambda_2-\lambda_1)]}{\lambda_2-\lambda_1} = 83$.

After cancellation, this becomes

$3[(\lambda_2^2+\lambda_1\lambda_2 + \lambda_1^2) + 3(\lambda_2 + \lambda_1) + 1] - \lambda_1\lambda_2[(\lambda_2 + \lambda_1) + 3] = 83$,

which can be written as

$3[(\lambda_2^2+2\lambda_1\lambda_2 + \lambda_1^2) - \lambda_1\lambda_2 + 3(\lambda_2 + \lambda_1) + 1] - \lambda_1\lambda_2[(\lambda_2 + \lambda_1) + 3] = 83$,

or equivalently, $3[(\lambda_2 + \lambda_1)^2 - \lambda_1\lambda_2 + 3(\lambda_2 + \lambda_1) + 1] - \lambda_1\lambda_2[(\lambda_2 + \lambda_1) + 3] = 83$.

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From $3[(\lambda_2+\lambda_1)+1]-\lambda_1\lambda_2=14$, we get $\lambda_1\lambda_2=3(\lambda_2+\lambda_1)-11$, and substituting this in the last equation, and denoting $\lambda_1+\lambda_2$ by x results in the equation $3(x^2+11-3x+3x+1)-(3x-11)(x+3)=83$, from which we get $x=7$.

Therefore, $\lambda_1+\lambda_2=7$ and $\lambda_1\lambda_2=3(7)-11=10$.
Then, $\lambda_1(7-\lambda_1)=10$, and solving the quadratic results in $\lambda_1=2$ or 5 .
These are the values of λ_1 and λ_2 . Let us assume that $\lambda_1=2$ and $\lambda_2=5$.
Then $2a+5(1-a)=3$ from which we get $a=\frac{2}{3}$ and $1-a=\frac{1}{3}$.
If we had assume that $\lambda_1=5$ and $\lambda_2=2$, then $a=\frac{1}{3}$ and $1-a=\frac{2}{3}$.