EXAM C QUESTIONS OF THE WEEK

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Question 6 - Week of August 29

You are given the following data sample of 1000 observations in interval grouped form:

Interval	Number of Observations in Interval
$(0, 100]$	275
$(100, 200]$	200
$(200, 500]$	280
$(500, 1000]$	150
$(1000, \infty)$	95

Three distributions are fitted to the data using maximum likelihood estimation, with the following outcomes.

Exponential distribution:
$\widehat{\theta} = 369.4$, maximum loglikelihood $= -676.9$

2-Parameter Pareto distribution:
$\widehat{\alpha} = 3.167$, $\widehat{\theta} = 901.2$, maximum loglikelihood $= -669.8$

Weibull distribution:
$\widehat{\tau} = .8550$, $\widehat{\theta} = 351.2$, maximum loglikelihood $= -671.4$.

(a) For each of the fitted models, apply the chi-square goodness-of-fit test to determine whether or not the model is an acceptable fit at a 5% level of significance.

(b) Determine which estimated model is preferable according to their chi-square p-values.

(c) Determine which estimated model is preferable according to the Schwarz Bayesian criterion.

(d) Test whether or not the Weibull model is preferable to the exponential model using the likelihood ratio test.

The solution can be found below.

Question 6 Solution

(a)&(b)

The chi-square statistic is $Q=\sum_{i=1}^{r}\frac{(E_i-O_i)^2}{E_i}$, where E_i is the expected number of observations for interval i.

Exponential: The cdf is $F(x)=1-e^{-x/\theta}$. Using $\widehat{\theta}=369.4$, we have

Interval	Estimated Prob.	Estimated Expected # of Obs.
$(0,100]$	$\widehat{F}(100)=1-e^{-100/369.4}=.237$	237
$(100,200]$	$\widehat{F}(200)-\widehat{F}(100)=.181$	181
$(200,500]$	$\widehat{F}(500)-\widehat{F}(200)=.324$	324
$(200,1000]$	$\widehat{F}(1000)-\widehat{F}(500)=.192$	192
$(1000,\infty)$	$1-\widehat{F}(1000)=.067$	67

$Q=\frac{(237-275)^2}{237}+\frac{(181-200)^2}{181}+\frac{(324-280)^2}{324}+\frac{(192-150)^2}{192}+\frac{(67-95)^2}{67}=35.0$

There are 5 intervals upon which the estimation is based, and one parameter is estimated in the exponential distribution, so the chi-square test has $5-1-1=3$ degrees of freedom.

From the chi-square table, we see that 99.5 percentile is 12.838, so the hypothesis that the exponential provides a good fit is rejected at the .5% level of significance (and any higher level of significance). The p-value of $Q=35.0$ is extremely small (the p-value for $Q=35.0$ is the probability that the chi-square with 3 degrees of freedom is greater than 35.0).

Pareto: The cdf is $F(x)=1-\left(\frac{\theta}{x+\theta}\right)^{\alpha}$. Using $\widehat{\alpha}=3.167$, $\widehat{\theta}=901.2$, we have

Interval	Estimated Prob.	Estimated Expected # of Obs.
$(0,100]$	$\widehat{F}(100)=1-\left(\frac{901.2}{100+901.2}\right)^{3.167}=.283$	283
$(100,200]$	$\widehat{F}(200)-\widehat{F}(100)=.187$	187
$(200,500]$	$\widehat{F}(500)-\widehat{F}(200)=.283$	283
$(200,1000]$	$\widehat{F}(1000)-\widehat{F}(500)=.153$	153
$(1000,\infty)$	$1-\widehat{F}(1000)=.094$	94

$Q=\frac{(283-275)^2}{283}+\frac{(187-200)^2}{187}+\frac{(283-280)^2}{283}+\frac{(153-150)^2}{153}+\frac{(94-95)^2}{94}=1.26$

Since there are 2 parameters estimated, there are $5-1-2=2$ degrees of freedom in the chi-square test. The 90th percentile of the chi-square with 2 degrees of freedom is 4.605, so the hypothesis of a good fit is not rejected at the 10% level of significance or lower. We have limited information about the chi-square distribution from the table, so we can determine with an good accuracy the p-value of $Q=1.26$ (a guess might be that it is in the 50% range; it must be between 10% and 95% by comparing Q to percentile values in the chi-square table).

Weibull: The cdf is $F(x)=1-e^{-(x/\theta)^{\tau}}$. Using $\widehat{\tau}=.855$, $\widehat{\theta}=351.2$, we have

Interval	Estimated Prob.	Estimated Expected # of Obs.
$(0,100]$	$\widehat{F}(100)=1-e^{-(100/351.2)^{.855}}=.289$	289
$(100,200]$	$\widehat{F}(200)-\widehat{F}(100)=.171$	172
$(200,500]$	$\widehat{F}(500)-\widehat{F}(200)=.280$	280
$(200,1000]$	$\widehat{F}(1000)-\widehat{F}(500)=.172$	172
$(1000,\infty)$	$1-\widehat{F}(1000)=.087$	87

$Q=\frac{(289-275)^2}{289}+\frac{(172-200)^2}{172}+\frac{(280-280)^2}{280}+\frac{(172-150)^2}{172}+\frac{(87-95)^2}{87}=8.63$

AS in the Pareto case, there are 2 parameters estimated and there are $5-1-2=2$ degrees of freedom in the chi-square test. The 95th percentile of the chi-square with 2 degrees of freedom is 45.991, so the hypothesis of a good fit is rejected at the 5% level of significance. From the chi-square table, we also see that with 2 df, the 97.5 percentile is 7.378 and the 99th percentile is 9.210 . $Q=8.63$ is between those values, so that the p-value of Q is between 2.5% and 1%.

The p-values of the three chi-square statistics are

Exponential	Pareto	Weibull
extremely small	about 50%	between 1% and 2.5%

The Pareto would be the preferred model, with Weibull next and exponential last according to p-values.

(c) To apply the Schwarz Bayesian criterion, for each estimated model we calculate $lnL - \frac{r}{2} ln\, n$, where L is the maximized likelihood, r is the number of model parameters estimated, and n is the number of data points upon which the maximum likelihood estimation was based.

Exponential: $-676.9 - \frac{1}{2}\, ln\, 1000 = -680.4$

Pareto: $-669.8 - \frac{2}{2}\, ln\, 1000 = -676.7$

Weibull: $-671.4 - \frac{2}{2}\, ln\, 1000 = -678.3$

Pareto is most preferable, Weibull is next and exponential is last.

(d) The test statistic for the likelihood ratio test is $2(ln\, L_A - ln\, L_B)$, where A is the estimated model with more parameters than B. In this case, A is Weibull and B is exponential. The test statistic is $2[-671.4 - (-676.9)] = 11.0$. The null hypothesis being tested is that the exponential model is "good enough" and the more sophisticated Weibull model is not significantly better. This is a chi-square test, where the number of degrees of freedom is the number of estimated parameters in model A minus the number of estimated parameters in model B. In this case, that is 1. From the chi-square table with 1 degree of freedom, we see that the 99.5 percentile is 7.879 . The null hypothesis is rejected at the .5% significance level.