EXAM P QUESTIONS OF THE WEEK

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Question 5 - Week of August 22

Smith is a quality control analyst who uses the exponential distribution with a mean of 10 years as the model for the exact time until failure for a particular machine. Smith is really only interested in the integer number of years, say X, until the machine fails, so if failure is within the first year, Smith regards that as 0 (integer) years until failure, and if the machine does not fail during the first year but fails in the second year, the Smith regards that as 1 (integer) year until failure, etc. Smith's colleague Jones, who is also a quality control analyst reviews Smith's model for the random variable X and has two comments:
(i) X has a geometric distribution, and
(ii) the mean of X is 10.

Determine which, if any, of the statements made by Jones are true?

The solution can be found below.

Question 5 Solution

We denote by W the exponential distribution with a mean of 10,
so the pdf of W is $f_W(w) = \frac{e^{-w/10}}{10}$.
Then the distribution of X can be found from the distribution of W.
X is a discrete integer-valued random variable ≥ 0.

$X = 0$ if $0 < W \leq 1$ (if failure is in the first year), and the probability is
$P(X = 0) = P(0 < W \leq 1) = \int_0^1 f_W(w)\, dw = \int_0^1 .1e^{-.1w}\, dw = 1 - e^{-.1} = .095163$.

$X = 1$ if $1 < W \leq 2$ (if failure is in the second year), and the probability is
$P(X = 1) = P(1 < W \leq 2) = \int_1^2 .1e^{-.1w}\, dw = e^{-.1} - e^{-.2} = .086107$.

$X = k$ if $k < W \leq k + 1$ (if failure is in the first year), and the probability is
$P(X = k) = P(k < W \leq k + 1) = \int_k^{k+1} .1e^{-.1w}\, dw$
$= e^{-.1k} - e^{-.1(k+1)} = (e^{-.1})^k (1 - e^{-.1})$.

The commonly used definition of the geometric distribution is as follows.
Suppose that $0 < p < 1$ and suppose that Z is an integer-valued random variable ≥ 0
with probability function $P(Z = j) = (1 - p)^k \cdot p$ for $j = 0, 1, 2, ...$.
Z is said to have a geometric distribution with parameter p, and the mean of Z is
$E[Z] = \frac{1-p}{p}$ (and the variance of Z is $Var[Z] = \frac{1-p}{p^2}$).

From the probability function of X described above, if we let $p = 1 - e^{-.1}$, then
$1 - p = e^{-.1}$ and $P(X = k) = (e^{-.1})^k (1 - e^{-.1}) = (1 - p)^k \cdot p$.
We see that X has a geometric distribution, with $p = 1 - e^{-.1}$.
The mean of X is $\frac{1-p}{p} = \frac{e^{-.1}}{1-e^{-.1}} = 9.508$.

Jones is correct about the distribution of X being geometric, but is wrong about the mean of X.