EXAM C QUESTIONS OF THE WEEK

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Question 5 - Week of August 22

Maximum likelihood estimation is applied to estimate the parameter θ in a Weibull random variable X for which $\tau = 3$ is known. The estimation is based on the following random sample of 6 observations: $3 \ , \ 4 \ , \ 6 \ , \ 6 \ , \ 7 \ , \ 9$

(a) Find the maximum likelihood estimate of θ.

(b) Using the observed information approach (see page 335 of the Loss Models text, 2nd ed.), find the estimated variance of the mle estimator of θ.

(c) Using the estimated variance from (b), find the estimated variance and approximate 95% confidence intervals for each of the following two quantities:
(i) $P[X \leq 5]$ and (ii) the median of X .

The solution can be found below.

Question 5 Solution

(a) The pdf of X is $f(x) = \frac{\tau x^{\tau-1} e^{-(x/\theta)^\tau}}{\theta^\tau}$.

For the random sample $x_1, x_2, ..., x_n$, the likelihood function is $L(\tau, \theta) = \frac{\tau^n \cdot \prod_{i=1}^{n} (x_i^{\tau-1}) \cdot e^{-\frac{1}{\theta^\tau} \cdot \Sigma x_i^\tau}}{\theta^{n\tau}}$,

and the loglikelihood function $l(\tau, \theta) = n\, ln(\tau) + (\tau - 1) \cdot \sum_{i=1}^{n} ln(x_i) - \frac{1}{\theta^\tau} \cdot \sum_{i=1}^{n} x_i^\tau - n\tau\, ln(\theta)$.

If τ is given then the mle equation is $\frac{d}{d\theta}\, l(\theta) = \frac{\tau}{\theta^{\tau+1}} \cdot \sum_{i=1}^{n} x_i^\tau - \frac{n\tau}{\theta} = 0$,

from which we get the mle $\widehat{\theta} = (\frac{1}{n} \cdot \sum_{i=1}^{n} x_i^\tau)^{1/\tau}$.

In this example we have $n = 6$ sample values and $\tau = 3$, so we get $\widehat{\theta} = (\frac{1}{6} \cdot \sum_{i=1}^{n} x_i^3)^{1/3} = 6.43$.

(b) The estimated variance of the mle $\widehat{\theta}$ is $\frac{1}{I(\theta)}$, where $I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2} \ell(\theta)\right]$ is the "information".

Continuing from part (a), we have

$$\frac{\partial^2}{\partial\theta^2} \ell(\theta) = \frac{\partial}{\partial\theta} [\frac{\tau}{\theta^{\tau+1}} \cdot \sum_{i=1}^{n} x_i^\tau - \frac{n\tau}{\theta}] = \frac{\partial}{\partial\theta} [\frac{3}{\theta^4} \cdot \sum_{i=1}^{n} x_i^3 - \frac{6\times 3}{\theta}] = -\frac{12}{\theta^5} \cdot \sum_{i=1}^{n} x_i^3 + \frac{18}{\theta^2} .$$

The expectation $E\left[\frac{\partial^2}{\partial\theta^2} \ell(\theta)\right]$ is equal to $E\left[-\frac{12}{\theta^5} \cdot \sum_{i=1}^{n} x_i^3 + \frac{18}{\theta^2}\right] = -\frac{12}{\theta^5} \cdot E\left[\sum_{i=1}^{n} x_i^3\right] + \frac{18}{\theta^2}$.

In order to find this expectation, we need to find $E\left[\sum_{i=1}^{n} x_i^3\right]$. According to the observed information approach, we use the observed sample values to find $E[x^3]$. This would be $\frac{3^3+4^3+6^3+6^3+7^3+9^3}{6} = \frac{1595}{6}$.

Then, $E\left[\sum_{i=1}^{n} x_i^3\right] = E[x_1^3 + \cdots + x_6^3]$ would be estimated to be $6 \times \frac{1595}{6} = 1595$. Using the mle $\widehat{\theta} = 6.43$ from part (a), we estimate $I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2} \ell(\theta)\right]$ to be

$$-[-\frac{12}{\widehat{\theta}^5} \cdot E\left[\sum_{i=1}^{n} x_i^3\right] + \frac{18}{\widehat{\theta}^2}] = -[-\frac{12}{6.43^5} \times 1595 + \frac{18}{6.43^2}] = 1.3 .$$

The estimated variance of $\widehat{\theta}$ is then $\frac{1}{1.3} = .77$.

(c) (i) For the Weibull distribution, $P[X \leq r] = 1 - e^{-(r/\theta)^\tau}$.

With $\tau = 3$, we have $P[X \leq 5] = 1 - e^{-(5/\theta)^3}$, and the mle estimate of this probability is

$1 - e^{-(5/\widehat{\theta})^3} = 1 - e^{-(5/6.43)^3} = .38$.

We use the following rule to find the variance of the estimate probability.

If $g(\widehat{\theta})$ is a function of the estimator $\widehat{\theta}$, then the variance of $g(\widehat{\theta})$ is approximately

$Var[\, g(\widehat{\theta})\,] = [g'(\widehat{\theta})]^2 \cdot Var[\,\widehat{\theta}\,]$ (this is the delta method).

In this case, $g(\widehat{\theta}) = 1 - e^{-(5/\widehat{\theta})^3}$, so that $g'(\widehat{\theta}) = -e^{-(5/\widehat{\theta})^3} \cdot \frac{375}{\widehat{\theta}^4}$.

Using the mle estimated value of $\widehat{\theta} = 6.43$, we get $g'(\widehat{\theta}) = -.137$.

Then, $Var[\, g(\widehat{\theta})\,]$ is approximately $(-.137)^2(.77) = .0145$.

The approximate 95% confidence interval for the estimated probability is

$.38 \pm 1.96\sqrt{.0145} = [.14\,,\,.62]$.

(ii) The median of the Weibull distribution is m, where $P[X \leq m] = 1 - e^{-(m/\theta)^\tau} = .5$.

Solving for m results in $m = \theta \cdot [\,ln\, 2\,]^{1/\tau}$.

In this example, with $\tau = 3$, we get $m = \theta \cdot [\,ln\, 2\,]^{1/3} = .885\,\theta$.

The mle estimate of the median will be $\widehat{m} = .885\,\widehat{\theta} = 5.69$.

The variance of the estimated median can be found as follows.

$\widehat{m} = .885\,\widehat{\theta} = g(\widehat{\theta})$, so that $g'(\widehat{\theta}) = .885$.

Then the approximate variance of $g(\widehat{\theta})$ is $[g'(\widehat{\theta})]^2 \cdot Var(\widehat{\theta}) = (.885)^2(.77) = .60$.

The approximate 95% confidence interval for the median is

$5.69 \pm 1.96\sqrt{.60} = [4.17\,,\,7.21]$.