EXAM C QUESTIONS OF THE WEEK

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Week of August 21/06

An inverse gamma distribution is fit to a data set using maximum likelihood estimation.
The estimates of α and θ are $\widehat{\alpha} = 4.3$ and $\widehat{\theta} = 21.1$.

The information matrix that results from the estimation of α and θ is $\begin{bmatrix} 4.0635 & .3175 \\ .3175 & 1.5873 \end{bmatrix}$.

Apply the delta method to find a 95% confidence interval for the mean of the distribution.

Solution can be found below.

Week of August 21/06 - Solution

The covariance matrix is the inverse of the information matrix. This will be

$$Cov(\widehat{\alpha},\widehat{\theta}) = \begin{bmatrix} 4.0635 & .3175 \\ .3175 & 1.5873 \end{bmatrix}^{-1} = \frac{1}{(1.5873)(4.0635)-(.3175)(.3175)} \cdot \begin{bmatrix} 1.5873 & -.3175 \\ -.3175 & 4.0635 \end{bmatrix}$$
$$= \begin{bmatrix} .2500 & -.0500 \\ -.0500 & .6400 \end{bmatrix} = \begin{bmatrix} V\widehat{ar}(\widehat{\alpha}) & C\widehat{ov}(\widehat{\alpha},\widehat{\theta}) \\ C\widehat{ov}(\widehat{\alpha},\widehat{\theta}) & V\widehat{ar}(\widehat{\theta}) \end{bmatrix}.$$

The mean of the inverse gamma is $\frac{\theta}{\alpha-1}$. The estimate of this is $\frac{21.1}{4.3-1} = 6.394$.

According to the delta method, the variance of the mle estimate of $g(\alpha,\theta) = \frac{\theta}{\alpha-1}$ is
$\left(\frac{\partial}{\partial\alpha}\, g(\alpha,\theta)\right)^2 \cdot Var(\widehat{\alpha}) + 2\left(\frac{\partial}{\partial\alpha}\, g(\alpha,\theta)\right)\left(\frac{\partial}{\partial\theta}\, g(\alpha,\theta)\right) \cdot Cov(\widehat{\alpha},\widehat{\theta}) + \left(\frac{\partial}{\partial\theta}\, g(\alpha,\theta)\right)^2 \cdot Var(\widehat{\theta})$
evaluated at the estimate values. This is

$$\left(-\frac{\theta}{(\alpha-1)^2}\right)^2(.25) + 2\left(-\frac{\theta}{(\alpha-1)^2}\right)\left(\frac{1}{\alpha-1}\right)(-.05) + \left(\frac{1}{\alpha-1}\right)^2(.64)$$
$$= \left(-\frac{21.1}{(4.3-1)^2}\right)^2(.25) + 2\left(-\frac{21.1}{(4.3-1)^2}\right)\left(\frac{1}{4.3-1}\right)(-.05) + \left(\frac{1}{4.3-1}\right)^2(.64) = 1.056\,.$$

The 95% confidence interval for the mean is $6.394 \pm 1.96\sqrt{1.056} = (4.38, 8.41)$.