# EXAM C QUESTIONS OF THE WEEK

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## Question 4 - Week of August 15

A two-parameter Pareto distribution for a ground up loss distribution has $\theta = 20$ . The parameter $\alpha$ is estimated using maximum likelihood estimation based on the following sample of 8 insurance payments:

$3\ ,\ 4\ ,\ 6\ ,\ 9\ ,\ 10\ ,\ 11\ ,\ 13\ ,\ 13$

The same sample is used in four separate estimations, each of which depends on a different interpretation of the sample. Find the maximum likelihood estimate of $\alpha$ based on each of the following interpretations.

(a) The sample is assumed to be a random sample of insurance payments, and the policy has no deductible and no policy limit.

(b) The sample is a random sample of insurance payments, and the policy has a limit of 13 per loss, and the two payments of 13 are limit payments.

(c) The sample is a random of insurance payments from a policy with an ordinary deductible of 3 and no policy limit.

(d) The sample is a random of insurance payments from a policy with an ordinary deductible of 3 and a maximum covered loss of 16, and the two payments of 13 are limit payments.

## Question 4 Solution

The pdf of the two-parameter Pareto is $f(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}$ and the cdf is $F(x) = 1 - (\frac{\theta}{x+\theta})^\alpha$ .

(a) Since all observations are exact ground up loss amounts without deductible or policy limit, the likelihood function is $L(\alpha) = \prod_{i=1}^{8} \frac{\alpha\theta^\alpha}{(x_i+\theta)^{\alpha+1}}$ , and the loglikelihood function is

$\ell(\alpha) = ln\, L(\alpha) = 8\, ln\, \alpha \,+\, 8\alpha\, ln\, \theta \,-\, (\alpha+1)\sum_{i=1}^{8} ln(x_i+\theta)$ .

Then $\frac{d}{d\alpha}\,\ell(\alpha) = \frac{8}{\alpha} \,+\, 8\, ln\, \theta \,-\, \sum_{i=1}^{8} ln(x_i+\theta) = 0$ results in an mle for $\alpha$ of

$\widehat{\alpha} = \frac{8}{\sum_{i=1}^{8} ln(x_i+\theta) - 8\, ln\, \theta} = \frac{8}{ln(23)+ln(24)+\cdots+ln(33)+ln(33)-8\, ln\, 20} = 2.86$ .

(b) The two insurance payments of 13 are limit payments and the first 6 payments are known loss amounts. All are ground up losses. The likelihood function is

$L(\alpha) = \prod_{i=1}^{6} f(x_i) \cdot [1-F(13)]^2 = \prod_{i=1}^{6} \frac{\alpha\theta^\alpha}{(x_i+\theta)^{\alpha+1}} \cdot [(\frac{\theta}{13+\theta})^\alpha]^2$ , and the loglikelihood function is

$\ell(\alpha) = ln\, L(\alpha) = 6\, ln\, \alpha \,+\, 6\alpha\, ln\, \theta \,-\, (\alpha+1)\sum_{i=1}^{6} ln(x_i+\theta) \,+ 2\alpha\, ln\, \theta - 2\alpha\, ln(13+\theta)$ .

Then $\frac{d}{d\alpha}\,\ell(\alpha) = \frac{6}{\alpha} \,+\, 8\, ln\, \theta \,-\, \sum_{i=1}^{6} ln(x_i+\theta) - 2\, ln(13+\theta) = 0$ results in an mle for $\alpha$ of

$\widehat{\alpha} = \frac{8}{\sum_{i=1}^{8} ln(x_i+\theta) - 8\, ln\, \theta} = \frac{6}{ln(23)+ln(24)+\cdots+ln(33)-8\, ln\, 20} = 2.14$ .

Note that the denominator is the same in (a) and (b), but in (b) the numerator is the number of non-limit payments.

(c) The likelihood function is constructed with conditional densities, $f(x|X>3)=\frac{f(x)}{1-F(3)}$ , where $x$ is the ground up loss (not payment) amount. Since we are given insurance payment amounts, the corresponding loss amounts are found by adding the deductible to the payment. The loss amounts are $6$ , $7$ , $9$ , $12$ , $13$ , $14$ , $16$ , $16$ . The likelihood function is

$L(\alpha)=\prod_{i=1}^{8}\left[\frac{f(x_i)}{1-F(3)}\right]=\prod_{i=1}^{8}\left[\frac{\alpha\theta^{\alpha}}{(x_i+\theta)^{\alpha+1}}\Big/(\frac{\theta}{3+\theta})^{\alpha}\right]=\frac{\alpha^{8}(3+\theta)^{8\alpha}}{\prod_{i=1}^{8}(x_i+\theta)^{\alpha+1}}$ , and the loglikelihood function is

$\ell(\alpha)=8\,ln\,\alpha\,+\,8\alpha\,ln(3+\theta)-(\alpha+1)\sum_{i=1}^{8}ln(x_i+\theta)$ .

Then $\frac{d}{d\alpha}\,\ell(\alpha)=\frac{8}{\alpha}+8\,ln(3+\theta)-\sum_{i=1}^{8}ln(x_i+\theta)\;=\;0$ results in an mle of

$\widehat{\alpha}=\frac{8}{ln(26)+ln(27)+\cdots+ln(36)+ln(36)-8\,ln(23)}=3.21$ .

(d) Since the deductible is 3 and the maximum covered loss is 16, the actual first six ground up loss amounts are $6$ , $7$ , $9$ , $12$ , $13$ , $14$ and the two limit payments are based on losses greater than 16. The likelihood function is

$$L=\prod_{i=1}^{8}\left[\frac{f(x_i)}{1-F(3)}\right]\times\left[\frac{1-F(16)}{1-F(3)}\right]^{2}=\prod_{i=1}^{6}\left[\frac{\alpha\theta^{\alpha}}{(x_i+\theta)^{\alpha+1}}\Big/(\frac{\theta}{3+\theta})^{\alpha}\right]\times\left[(\frac{\theta}{16+\theta})^{\alpha}\Big/(\frac{\theta}{3+\theta})^{\alpha}\right]^{2}$$

$$=\frac{\alpha^{6}(3+\theta)^{8\alpha}}{\left[\prod_{i=1}^{6}(x_i+\theta)^{\alpha+1}\right]\times(16+\theta)^{2\alpha}}\ .$$

Loglikelihood function is

$\ell(\alpha)=6\,ln\,\alpha\,+\,8\alpha\,ln(3+\theta)-(\alpha+1)\sum_{i=1}^{6}ln(x_i+\theta)-2\alpha\,ln(16+\theta)$ .

Then $\frac{d}{d\alpha}\,\ell(\alpha)=\frac{6}{\alpha}+8\,ln(3+\theta)-\sum_{i=1}^{6}ln(x_i+\theta)\;-2\,ln(16+\theta)=\;0$ results in an mle of

$\widehat{\alpha}=\frac{6}{ln(26)+ln(27)+\cdots+ln(36)+ln(36)-8\,ln(23)}=2.41$ .

Note that the denominator in (d) is the same as that in (c), but the numerator in (d) is 6, the number of non-limit payments.