EXAM P QUESTIONS OF THE WEEK

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An insurance claims administrator verifies claims for various loss amounts.
For a loss claim of amount x, the amount of time spent by the administrator to verify the claim is uniformly distributed between 0 and $1+x$ hours. The amount of each claim received by the administrator is uniformly distributed between 1 and 2. Find the average amount of time that an administrator spends on a randomly arriving claim.

The solution can be found below.

Week of April 10/06 - Solution

$X =$ amount of loss claim, uniformly distributed on $(1,2)$, so $f_X(x) = 1$ for $1 < x < 2$.
$Y =$ amount of time spent verifying claim.
We are given that the conditional distribution of Y given $X = x$ is uniform on $(0, 1+x)$, so $f(y|x) = \frac{1}{1+x}$ for $0 < y < 1+x$.

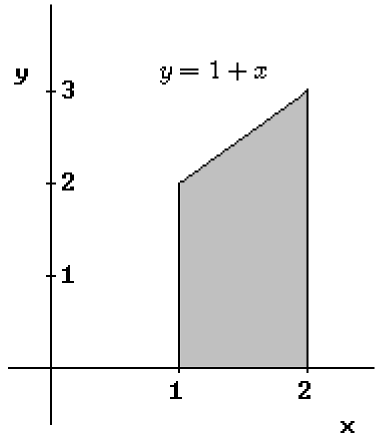
We wish to find $E[Y]$. The joint density of X and Y is
$f(x,y) = f(y|x) \cdot f_X(x) = \frac{1}{1+x}$ for $0 < y < 1+x$ and $1 < x < 2$.
There are a couple of ways to find $E[Y]$:

(i) $E[Y] = \int\int y\, f(x,y)\, dy\, dx$ or $E[Y] = \int\int y\, f(x,y)\, dx\, dy$, with careful setting of the integral limits, or
(ii) $E[Y] = \int y\, f_Y(y)\, dy$, where $f_Y(y)$ is the pdf of the marginal distribution of Y.
$E[Y] = \int_1^2 \int_0^{1+x} y \cdot$
(iii) The double expectation rule, $E[Y] = E[\,E[Y|X]\,]$.

If we apply the first approach for method (i), we get
$E[Y] = \int_1^2 \int_0^{1+x} y \cdot \frac{1}{1+x}\, dy\, dx = \int_1^2 \frac{(1+x)^2}{2(1+x)}\, dy = \int_1^2 \frac{1+x}{2}\, dx = \frac{5}{4}$.

If we apply the second approach for method (i), we must split the double integral into
$E[Y] = \int_0^2 \int_1^2 y \cdot \frac{1}{1+x}\, dx\, dy + \int_2^3 \int_{y-1}^2 y \cdot \frac{1}{1+x}\, dx\, dy$
The first integral becomes $\int_0^2 y\, ln(\frac{3}{2})\, dy = 2\, ln(\frac{3}{2})$.
The second integral becomes $\int_2^3 y\, [ln3 - ln\, y]\, dy = \frac{5}{2}\, ln3 - \int_2^3 y\, ln\, y\, dy$.
The integral $\int_2^3 y\, ln\, y\, dy$ is found by integration by parts.
Let $\int y\, ln\, y\, dy = A$.
Let $u = y$ and $dv = ln\, y\, dy$, then $v = y\, lny - y$ (antiderivative of $ln\, y$), and then
$A = \int y\, ln\, y\, dy = y(y\, lny - y) - \int (y\, ln\, y - y)\, dy = y^2\, ln\, y - y^2 - A + \frac{y^2}{2}$,
so that $A = \int y\, ln\, y\, dy = \frac{1}{2} y^2\, ln\, y - \frac{y^2}{4}$.
Then $\int_2^3 y\, ln\, y\, dy = \frac{1}{2} y^2\, ln\, y - \frac{y^2}{4} \Big|_2^3 = \frac{9}{2}\, ln3 - \frac{9}{4} - (\frac{4}{2}\, ln\, 2 - 1) = \frac{9}{2}\, ln3 - 2\, ln2 - \frac{5}{4}$.
Finally, $E[Y] = 2\, ln(\frac{3}{2}) + \frac{5}{2}\, ln3 - \int_2^3 y\, ln\, y\, dy$
$= 2\, ln\, 3 - 2\, ln\, 2 + \frac{5}{2}\, ln3 - (\frac{9}{2}\, ln3 - 2\, ln2 - \frac{5}{4}) = \frac{5}{4}$.
The first order of integration for method (i) was clearly the more efficient one.

(ii) This method is equivalent to the second approach in method (i), because we find $f_Y(y)$ from the relationship $f_Y(y) = \int f(x,y)\,dx$. The two-dimensional region of probability for the joint distribution is $1 < x < 2$ and $0 < y < 1+x$. This is illustrated in the graph below



For $0 < y < 2$, $f_Y(y) = \int_1^2 f(x,y)\,dx = \int_1^2 \frac{1}{1+x}\,dx = ln(\frac{3}{2})$
and for $2 \le x < 3$, $f_Y(y) = \int_{y-1}^2 f(x,y)\,dx = \int_{y-1}^2 \frac{1}{1+x}\,dx = ln\,3 \,-\, ln\,y$.
Then $E[Y] = \int_0^2 y\,ln(\frac{3}{2})\,dy \;+\; \int_2^3 y\,[ln3 \,-\, ln\,y]\,dy$, which is the same as the second part of method (i).

(iii) According to the double expectation rule, for any two random variables U and W, we have $E[U] = E[\,E[U|W]\,]$. Therefore, $E[Y] \,=\, E[\,E[Y|X]\,]$.
We are told that the conditional distribution of Y given $X = x$ is uniform on the interval $(0, 1+x)$, so $E[Y|X] = \frac{1+X}{2}$. Then $E[\,E[Y|X]\,] = E[\frac{1+X}{2}] = \frac{1}{2} + \frac{1}{2}E[X] = \frac{1}{2} + \frac{1}{2}(\frac{3}{2}) = \frac{5}{4}$, since X is uniform on $(1,2)$ and X has mean $\frac{3}{2}$.