SPRING 2008 CAS COURSE 3L SOLUTIONS

1. Let K be the number of customers who buy a policy. We want the probability $P(K \geq 56)$.
With the integer correction this becomes $P(K \geq 55.5)$.
To apply the central limit theorem we need $E(K)$ and $Var(K)$.
K has a binomial distribution with $n = 500$ and $q = .10$ (success probability).
Therefore, $E(K) = 500(.1) = 50$ and $Var(K) = 500(.1)(.9) = 45$.
Then, $P(K \geq 55.5) = P(\frac{K-50}{\sqrt{45}} \geq \frac{55.5-50}{\sqrt{45}}) = P(Z \geq .82) = 1 - \Phi(.82) = 1 - .7939 = .2061$.
Answer: B

2. I. False. $E(\widehat{\theta}_n) = E(5,000n/(n+1)) = 5000n/(n+1) \neq 5000 = \theta$.
II. True. $|\theta - \widehat{\theta}_n| = |5000 - 5000n/(n+1)| = 5000/(n+1) \rightarrow 0$ as $n \rightarrow \infty$,
so that if $\delta > 0$, then $P(|\theta - \widehat{\theta}_n| < \delta) = P(\frac{5000}{n+1} < \delta) = 1$ when n is large.
Therefore, $\lim\limits_{n \rightarrow \infty} P(|\theta - \widehat{\theta}_n| < \delta) = 1$ for any $\delta > 0$, which means that the estimator is consistent.
III. True. $MSE(\widehat{\theta}_{10}) = E[(\widehat{\theta}_{10} - \theta)^2] = E[(\frac{5000(10)}{11} - 5000)^2] = 206,612$.
Answer: D

3. If X has an inverse exponential distribution parameter θ, then $Y = \frac{1}{X}$ has an exponential distribution with mean $\frac{1}{\theta}$. We transform the given inverse exponential data set of sample x's to sample y's with $y = \frac{1}{x}$. So $\frac{1}{8000}$, $\frac{1}{10,000}$, $\frac{1}{12,000}$, $\frac{1}{15,000}$ is a sample from an exponential distribution with mean $\frac{1}{\theta}$. The mle of $\frac{1}{\theta}$ is $\frac{\frac{1}{8000}+\frac{1}{10,000}+\frac{1}{12,000}+\frac{1}{15,000}}{4} = .00009375$.
The mle of θ is $\frac{1}{.00009375} = 10,667$. Answer: C

4. Assuming that $\lambda = .4$, we have
$P(\text{No Claims}) = e^{-.4} = .67032$, $P(\text{One Claims}) = .4e^{-.4} = .26813$,
and $P(\text{Two or More Claims}) = 1 - (.67032 + .26813) = .06155$.
Out of 100 insureds, based on the null hypothesis, the expected number of insureds with no claims is $100(.67032) = 67.03$, the expected number with one claim is 26.81, and the expected number of insureds with 2 two or more claims is 6.16. The chi square statistic is
$\frac{(67.03-74)^2}{67.03} + \frac{(26.81-16)^2}{26.81} + \frac{(6.16-10)2^2}{6.16} = 7.5$. Answer: D

5. The probability of Type I error is the probability of rejecting H_0 when it is true.
This is $P(X > k|\mu = 1)$. We are told that this is .025. Since X is normal, we have
$P(X > k|\mu = 1) = P(\frac{X-1}{1.5} > \frac{k-1}{1.5}|\mu = 1) = 1 - \Phi(\frac{k-1}{1.5}) = .025$.
It follows that $\frac{k-1}{1.5} = 1.96$, and therefore, $k = 3.94$.
The probability of Type II error is the probability of not rejecting H_0 when H_1 is true.
This is $P(X \leq k|\mu = 5) = P(\frac{X-5}{1.5} \leq \frac{3.94-5}{1.5}|\mu = 5) = \Phi(-.71) = 1 - \Phi(.71)$
$= 1 - .7611 = .24$. Answer: C

6. The standard deviation of the sample mean is $\frac{\text{sample standard deviation}}{\sqrt{25}} = 1600$
The test statistic is $|\frac{42{,}000-45{,}000}{1600}| = 1.875$.
Since this is a 2-sided test, we must consider both tails as the rejection region.
From the t-table with 24 degrees of freedom, we see that $P(|t| > 1.711) = .1$
and $P(|t| > 2.064) = .05$. Since the test statistic is between 1.711 and 2.064,
the result of the test would be to reject at the 10% level but not reject at the 5% level.
(Thanks to Abe Weishaus for reminding me that the t-distribution is appropriate here).
Answer: D

7. The probability of an individual claim exceeding 300 is $\frac{700-300}{700-50} = \frac{8}{13}$.
The number of claims N exceeding 300 in a random sample of 3 claims has a binomial distribution with $n = 3$ and $q = \frac{8}{13}$. The probability of no more than two claims exceeding 300 is $P(N \leq 2) = 1 - P(N = 3) = 1 - (\frac{8}{13})^3 = .77$.
(Thanks to nittanylions on the actuarial outpost for pointing out an error in the original solution I posted). Answer: E

8. The larger claim is Y_2 ($k-2$), the second order statistic of a sample of $n = 2$.
The pdf of Y_2 is $\frac{2!}{(2-1)!(2-2)!}[F(y)]^{2-1}[1-F(y)]^{2-2}f(y)$.
For the exponential distribution with mean 1000, $f(y) = .001e^{-.001y}$ and
$F(y) = 1 - e^{-.001y}$, so that the pdf of Y_2 is $2(1-e^{-.001y})(.001e^{-.001y})$.
The mean of Y_2 is
$\int_0^\infty y \cdot 2(1-e^{-.001y})(.001e^{-.001y})\,dy$
$= \int_0^\infty (2 \times .001ye^{-.001y} - .002ye^{-.002y})\,dy = 2 \times \frac{1}{.001} - \frac{1}{.002} = 1500.$
The integral is 2 times the mean of an exponential with mean 1000 minus the mean of an exponential with mean 500. Alternatively, we could use the rule $\int_0^\infty t^k e^{-at}\,dt = \frac{k!}{a^{k+1}}$
Answer: C

9. The regression line is $\widehat{Y} = \widehat{\alpha} + \widehat{\beta}X$ and the residual at X_i is $\widehat{\epsilon}_i = Y_i - \widehat{Y}_i$.
The residual at $X_3 =$ 6.39 will be $427.75 - \widehat{Y}_3 = 427.75 - (\widehat{\alpha} + 34.5 \times 6.39)$.
For a simple linear regression, we have $\overline{Y} = \widehat{\alpha} + \widehat{\beta}\overline{X}$. Therefore,
$\frac{2357.55}{5} = \widehat{\alpha} + 34.5 \times \frac{39.08}{5}$, from which we get $\widehat{\alpha} = 201.858$.
The residual is $427.75 - (201.858 + 34.5 \times 6.39) = 5.4$. Answer: D

10. Assuming that the number of accidents is independent from one day to another, since the sum of independent Poisson random variables is Poisson, the distribution of the total number of accidents in a week N is Poisson with a mean of 20 (sum of the daily means).
Then $P(N = 18) = \frac{e^{-20}20^{18}}{18!}$.
With some clever calculation sequences we can find this probability.
Find $20e^{-1}$ and sequentially multiply by itself and divide by 2, then 3, then 4, etc.
up to 18. This will result in $\frac{e^{-18}20^{18}}{18!}$, and then multiply that by e^{-2} to get .084 . Answer: D

11. The number of calls X received from 9AM to 1PM has a Poisson distribution with a mean of $3 \times 30 + 10 = 100$, and the number of calls Y received from 1PM to 5PM has a Poisson distribution with a mean of $2 \times 25 + 2 \times 30 = 110$. The probability that X exceeds Y is $P(X > Y) = P(X - Y > 0)$. The mean of $X - Y$ is $100 - 110 = -10$ and since X and Y are independent, the variance of $X - Y$ is $100 + 110 = 210$. Applying the normal approximation, with integer correction, to $W = X - Y$, we get $P(W > -.5) = P(\frac{W-(-10)}{\sqrt{210}} > \frac{-.5-(-10)}{\sqrt{210}}) = 1 - \Phi(.66) = 1 - .7454 = .25$.

Answer: C

12. The aggregate losses per year S for the insurance company follow a compound Poisson distribution with a mean of $E(S) = E(N) \times E(X) = 2 \times 10 = 20$ (million). Since S has a compound Poisson distribution, the variance of S is
$Var(S) = E(N) \times E(X^2) = 2 \times (2 \times 10^2) = 400$ (million^2).
The risk load for the company is $.1 \times (20 + \sqrt{400}) = 4$ (million).
The risk load per policy is $\frac{4{,}000{,}000}{1{,}000{,}000} = 4$. Answer: A

13. This is the survival function for the Generalized DeMoivre model. The complete expected future lifetime is $\overset{\circ}{e}_x = \frac{\omega - x}{\alpha + 1}$. In this case, that will be $\frac{40-30}{1.1+1} = 4.76$.
(Thanks to nittanylions on the actuarial outpost for pointing out an error in the original solution I posted). Answer: A

14. ${}_{2|2}q_{66} = {}_2p^*_{66} \cdot {}_2q^*_{68} = p_{66} \cdot p^*_{67} \cdot (1 - p^*_{68} \cdot p^*_{69})$.
From the given information, we have $p^{IT}_{66} = .97671$,
$p^*_{67} = 1 - q^*_{67} = 1 - 4(.02544) = .89824$,
$p^*_{68} = 1 - q^*_{68} = 1 - 4(.02779) = .88884$, and $p^*_{69} = p_{66} = .97671$.
Then ${}_{2|2}q_{66} = (.97671)(.89824)[1 - (.88884)(.97671)] = .1157$. Answer: B

15. The force of mortality can be found from the survival function:
$\mu(x) = -\frac{s'(x)}{s(x)} = -\frac{e^{-5x^7}\cdot(-35x^6)}{e^{-5x^7}} = 35x^6$.
Or we can use $\mu(x) = -\frac{d}{dx}\, ln\, s(x) = -\frac{d}{dx}\,(-5x^7) = 35x^6$
(thanks ro Abe Weishaus for suggesting the log approach). Answer: B

16. A general relationship under UDD is that ℓ_x is linear within each year of age .
In this problem we have
${}_{.5|.5}q_{45.25} = {}_{.5}p_{45.25} - p_{45.25} = \frac{\ell_{45.75}-\ell_{46.25}}{\ell_{45.25}} = \frac{925-860}{975} = .067$. Answer: D

17. The joint survival is $p_{2_x:2_y} = p_{2_x} \times p_{2_y}$.
The hazard rate is the force of mortality, so $p_{2_x} = e^{-\int_2^3 \lambda_x\, dx} = e^{-\int_2^3 kx^4\, dx} = e^{-42.2k}$,
and $p_{2_x} = e^{-\int_2^3 ky^4\, dy} = e^{-42.2k}$.
If $p_{2_x:2_y} = p_{2_x} \times p_{2_y} = e^{-42.2k} \times e^{-42.2k} = .5$, we solve for k to get $k = .00821$.
(Thanks to nittanylions on the actuarial outpost for pointing out an error in the original solution I posted). Answer: A

18. $S_{T(x)\,T(y)}(s,t) = P[(T(x) > s) \cap (T(y) > t)] = P[T(x) > s] \times P[T(y) > t]$
(the last equality follows from independence of $T(x)$ and $T(y)$).
$P[T(x) > s] = \int_s^\infty f(z)\,dz = \int_s^\infty \frac{1}{(1+z)^2}\,dz = \frac{1}{1+s}$,
and in a similar way, $P[T(y) > t] = \frac{1}{1+t}$.
Then, $S_{T(x)\,T(y)}(s,t) = \frac{1}{1+s} \times \frac{1}{1+t}$. Answer: A

19. $q_x^{(\tau)} = 1 - p_x^{(\tau)}$ and $p_x^{(\tau)} = e^{-\int_0^1 \mu_x^{(\tau)}(t)\,dt}$.
Since $\mu_x^{(\tau)}(t) = \mu_x^{(1)}(t) + \mu_x^{(2)}(t) + \mu_x^{(3)}(t) + \mu_x^{(4)}(t) = \frac{t^3+3t^2+3t+1}{100} = \frac{(t+1)^3}{100}$,
we get $\int_0^1 \mu_x^{(\tau)}(t)\,dt = \int_0^1 \frac{(t+1)^3}{100}\,dt = .0375$ and $q_x^{(\tau)} = 1 - e^{-.0375} = .0368$.
(Thanks to Feif on the actuarial outpost for pointing out an error in the original solution).
Answer: D

20. We wish to find ${}_2Q_0^{(2,2)}$. We can multiply the matrix by itself and get the 2,2 entry as ${}_2Q_0^{(2,2)} = (.2)(.5) + (.6)(.6) + (.2)(.3) = .52$. Answer: D

21. An exponential survival distribution with mean m corresponds to a constant force of mortality of $\mu = \frac{1}{m}$. Policy A has a constant force of mortality of $\mu = \frac{1}{5} = .2$.
Assuming that the benefit is paid at the moment of death, with a constant force of mortality we have $\overline{A}_x = \frac{\mu}{\mu+\delta} = \frac{.2}{.2+ln(1.06)} = .7744$. If the benefit is paid at the end of the year of death then $A_x = \frac{1-e^{-\mu}}{1-e^{-\mu}+i} = .7513$.
A uniform distribution for survival corresponds to DeMoivre's law. In this case, we have $\omega = 80$. Under DeMoivre's law, $\overline{A}_x = \frac{1}{\omega-x} \cdot \overline{a}_{\overline{\omega-x}|} = \frac{1}{80-70} \cdot \overline{a}_{\overline{10}|} = .7579$.
If the benefit is paid at the end of the year of death, the policy price is $A_x = \frac{1}{\omega-x} \cdot a_{\overline{\omega-x}|} = \frac{1}{80-70} \cdot a_{\overline{10}|} = .7360$.
The absolute difference in continuous policy prices is $.0165$ and the absolute difference in the discrete policy prices is .0153. Answer: B

22. The present value random variable for the benefit is $Z = e^{-.03T}$.
The 80th percentile of Z is c, where $P(Z \leq c) = .8$.
The inequality $Z \leq c$ is equivalent to $e^{-.03T} \leq c$, and equivalent to $T \geq \frac{ln\,c}{-.03}$.
Therefore, $P(T \geq \frac{ln\,c}{-.03}) = .8$. If we let $a = \frac{ln\,c}{-.03}$, then ${}_ap_x = .8$.
With constant force .06, we have ${}_ap_x = e^{-a\mu} = e^{-.06a} = .8$.
We also have $e^{-.06a} = (e^{-a})^{.06} = (e^{-\frac{ln\,c}{-.03}})^{.06} = c^2 = .8$.
Therefore, $c = .894$. Answer: E

23. From the Illustrative Table we have $1000A_{30} = 102.48$.
The APV of the premiums is $2x \cdot \ddot{a}_{30} - x \cdot \ddot{a}_{30:\overline{5}|}$.
From the Illustrative Table we have $\ddot{a}_{30} = 15.8561$,
and
$\ddot{a}_{30:\overline{5}|} = \ddot{a}_{30} - v^5\,{}_5p_{30}\,\ddot{a}_{35} = \ddot{a}_{30} - {}_5E_{30}\,\ddot{a}_{35}$
$= 15.8561 - (.74091)(15.3926) = 4.4516$.
The APV of premiums is $2x(15.8561) - x \cdot (4.4516) = 27.2606x$.
Then $27.2606x = 102.48$, so that $x = 3.76$. Answer: E

24. The reserve on an endowment insurance can be formulated in annuity form as
$_4V_{35:\overline{5}|} = 1 - \frac{\ddot{a}_{35:\overline{1}|}}{\ddot{a}_{35:\overline{5}|}} = 1 - \frac{1}{4.2} = .762$.
We are asked for $_4\overline{V}(\overline{A}_{35:\overline{5}|})$, but we would need an assumption such as UDD to find the continuous reserve. Answer: C

25. Since transitions are at the end of each year, the cash flow is 0 in the first year at time .5. There will be a cash flow of 30 at time 1.5 with probability $Q^{(1,2)} = .25$, and there will be a cash flow of 30 at time 2.5 with probability $_2Q^{(1,2)} = (.75)(.25) + (.25)(.5) = .3125$.
The APV of the cash flows is $30[.25v^{1.5} + .3125v^{2.5}] = 14.98$. Answer: B